

Homework #3 Solutions

$$\textcircled{1} \quad S_X = \{0, 1, 2, 3, 4\}$$
$$S_Y = \{0, 1\} \text{ and } \{Y=1\} \Leftrightarrow \{\lambda=4\}$$
$$S_Z = \{0, 1\} \text{ and } \{Z=0\} \Leftrightarrow \{X=0\}$$

$$\textcircled{a} \quad P(X=0, Y=0, Z=0) = \left(\frac{3}{4}\right)^4$$

$$P(X=1, Y=0, Z=1) = \binom{4}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^3$$

$$P(X=2, Y=0, Z=1) = \binom{4}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2$$

$$P(X=3, Y=0, Z=1) = \binom{4}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)$$

$$P(X=4, Y=1, Z=1) = \left(\frac{1}{4}\right)^4$$

= 0 otherwise

$$\text{(b)} \quad P(X=1, Y=0, Z=1) = 1$$

= 0 otherwise.

$$\textcircled{2} \quad \text{(a)} \quad P[|X| < 5, Y > 2, Z^2 \geq 2]$$

$$= P[|X| < 5] P[Y > 2] P[Z^2 \geq 2]$$

$$= P[-5 < X < 5] (1 - P[Y \leq 2]) (1 - P[\sqrt{2} < Z < \sqrt{2}])$$

$$= (F_X(5) - F_X(-5)) (1 - F_Y(2)) (1 - F_X(\sqrt{2}) + F_X(-\sqrt{2}))$$

$$\text{(b)} \quad P[X > 5, Y < 0, Z=1] = P[X > 5] P[Y < 0] P[Z=1]$$

$$= [1 - F_X(5)] F_Y(0^-) (F_Z(1) - F_Z(1^-))$$

$$\text{(c)} \quad P[\min(X, Y, Z) > 2] = P[X > 2, Y > 2, Z > 2]$$

$$= P[X > 2] P[Y > 2] P[Z > 2] = [1 - F_X(2)] [1 - F_Y(2)] [1 - F_Z(2)]$$

$$\text{(d)} \quad P[\max(X, Y, Z) < 6] = P[X < 6, Y < 6, Z < 6]$$

$$= P[X < 6] P[Y < 6] P[Z < 6] = F_X(6^-) F_Y(6^-) F_Z(6)$$

③ (a) All outcomes are equiprobable

$$P[X_1=i, X_2=j] = P[X_1=i]P[X_2=j] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} ; \begin{matrix} 1 \leq j \leq 6 \\ 1 \leq i \leq 6 \end{matrix}$$

(b) $X = \min(X_1, X_2)$ $Y = \max(X_1, X_2)$

$$\begin{aligned} P[X=i, Y=j] &= 0 && ; i > j \\ &= P[X_1=i, X_2=i] = \frac{1}{36} && ; i=j \\ &= P[X_1=i, X_2=j] + P[X_1=j, X_2=i] = \frac{2}{36} && ; i < j \end{aligned}$$

(c) $P[Y=1] = \frac{1}{36}$

$P[Y=2] = \frac{3}{36}$

$P[Y=3] = \frac{5}{36}$

$P[Y=4] = \frac{7}{36}$

$P[Y=5] = \frac{9}{36}$

$P[Y=6] = \frac{11}{36}$

$P[X=1] = \frac{11}{36}$

$P[X=2] = \frac{9}{36}$

$P[X=3] = \frac{7}{36}$

$P[X=4] = \frac{5}{36}$

$P[X=5] = \frac{3}{36}$

$P[X=6] = \frac{1}{36}$

④ (a) All three cases have the same marginal pmf

$$P[X=-1] = P[X=0] = P[X=1] = \frac{1}{3}$$

$$P[Y=-1] = P[Y=0] = P[Y=1] = \frac{1}{3}$$

(b)

	i	ii	iii
P(A)	2/3	2/3	2/3
P(B)	5/6	2/3	2/3
P(C)	2/3	1/3	1

⑤ (a) For $x > 0, y > 0$

$$F_{xy}(x, y) = \int_0^x \int_0^y a x e^{-ax^2/2} b y e^{-by^2/2} dx dy$$

$$= (1 - e^{-ax^2/2})(1 - e^{-by^2/2})$$

(b) $P[X > Y] = \int_0^{\infty} \int_0^x a x e^{-ax^2/2} b y e^{-by^2/2} dy dx$

$$= \int_0^{\infty} a x e^{-ax^2/2} (1 - e^{-bx^2/2}) dx = 1 - \frac{a}{a+b}$$

(c) $F_X(x) = \lim_{y \rightarrow \infty} F_{xy}(x, y) = 1 - e^{-ax^2/2} \quad x > 0$

$$\frac{dF_X(x)}{dx} = f_X(x) = a x e^{-ax^2/2} \quad x > 0$$

Similarly, $f_Y(y) = b y e^{-by^2/2}$

⑥ (a) $\int_0^1 \int_0^1 k(x+y) dx dy = 1$

$$\Rightarrow k(1) = 1 \Rightarrow k = 1$$

(b) For $0 < x < 1; 0 < y < 1$

$$F_{xy}(x, y) = \int_0^y \int_0^x x+y dy dx = \frac{xy(x+y)}{2}$$

For $x < 1; y > 1$

$$F_{xy}(x, y) = \int_0^1 \int_0^x x+y dx dy = \frac{x(x+1)}{2}$$

For $x > 1; y < 1$

$$F_{xy}(x, y) = \int_0^1 \int_0^y x+y dy dx = \frac{y(y+1)}{2}$$

For $x > 1; y > 1$

$$F_{xy}(x, y) = 1$$

$$(c) F_X(x) = \lim_{y \rightarrow \infty} F_{XY}(x, y) = F_{XY}(x, 1) \quad 0 < x < 1$$

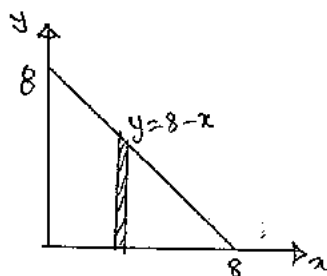
$$= \frac{x(x+1)}{2}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = x + \frac{1}{2}$$

Similarly $f_Y(y) = y + \frac{1}{2}$

(7) The probability is obtained by integrating the joint pdf over the region indicated:

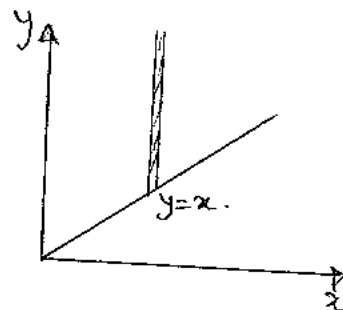
$$(a) P[X+Y \leq 8] = \int_0^8 \int_0^{8-x} 2e^{-x}e^{-2y} dy dx$$



$$= \int_0^8 e^{-x} (1 - e^{-2(8-x)}) dx = 1 - 2e^{-8} + e^{-16}$$

$$(b) P[X < Y] = \int_0^{\infty} \int_x^{\infty} 2e^{-x}e^{-2y} dy dx$$

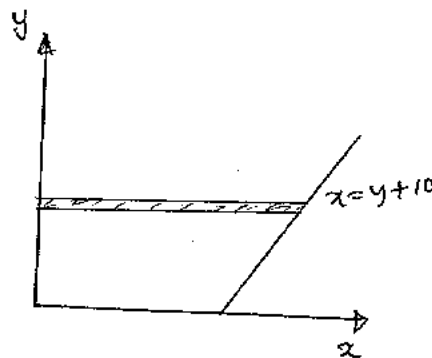
$$= \int_0^{\infty} e^{-x} e^{-2x} dx = \frac{1}{3}$$



$$(c) P[X-Y \leq 10] = \int_0^{\infty} \int_0^{y+10} e^{-x} dx 2e^{-2y} dy$$

$$= 2 \int_0^{\infty} (1 - e^{-(y+10)}) e^{-2y} dy$$

$$= 1 - \frac{2}{3} e^{-10}$$



$$(d) P[X^2 < Y] = \int_0^{\infty} \int_{x^2}^{\infty} e^{-x} 2 \cdot e^{-2y} dy dx$$

$$= \int_0^{\infty} e^{-x} e^{-2x^2} dx = e^{\frac{1}{8}} \int_0^{\infty} e^{-2(x^2 + \frac{1}{2}x + \frac{1}{16})} dx$$

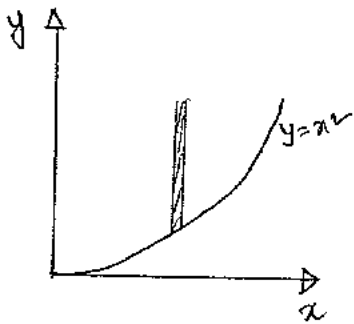
$$= e^{\frac{1}{8}} \sqrt{2\pi(\frac{1}{4})} \int_0^{\infty} \frac{e^{-(x+\frac{1}{4})^2 / 2(\frac{1}{4})}}{\sqrt{2\pi(\frac{1}{4})}} dx$$

$$= e^{\frac{1}{8}} \sqrt{\frac{\pi}{2}} \int_{1/2}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= e^{\frac{1}{8}} \sqrt{\frac{\pi}{2}} Q\left(\frac{1}{2}\right)$$

$$\text{let } z = \frac{x + 1/4}{1/2}$$

$$\text{where } Q(x) \triangleq \int_x^{\infty} \frac{e^{-\lambda^2/2}}{\sqrt{2\pi}} d\lambda$$



$$(8) f_X(x) = \int_0^{\infty} x e^{-x} e^{-xy} dy = x e^{-x} \left(\frac{-1}{x} e^{-xy} \right)_0^{\infty} = e^{-x}$$

$$f_Y(y) = \int_0^{\infty} x e^{-x(1+y)} dx = \frac{e^{-x(1+y)} [(1+y)x - 1]}{(1+y)^2} \Big|_0^{\infty} = \frac{1}{(1+y)^2}$$

NOTE: Questions 9 & 10 are mistyped. Refer to questions 23, 24 pg 259

$$(9) (a) P[a < X \leq b] = P[a < X \leq b] P[Y \leq d] = [F_X(b) - F_X(a)] F_Y(d)$$

$$(b) P[a \leq X \leq b, c \leq Y \leq d] = [F_X(b) - F_X(a-)] [F_Y(d) - F_Y(c-)]$$

$$(c) P[|X| > a, c \leq Y \leq d] = (1 - F_X(a) + F_X(a-)) [F_Y(d) - F_Y(c-)]$$

$$(10) (a) P[X^2 < 1/2, |Y-1| < 1/2] = P[X^2 < 1/2] P[|Y-1| < 1/2]$$

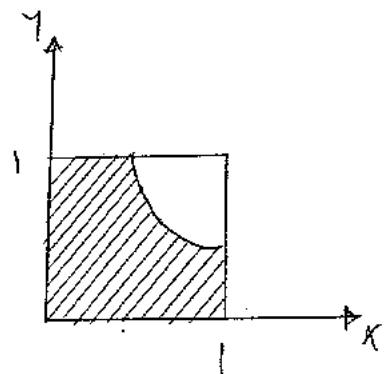
$$= P[X < \frac{1}{\sqrt{2}}] P[Y > \frac{1}{2}] = \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$(b) P[X/2 < 1, Y > 0] = P[X < 2] P[Y > 0] = 1$$

$$(c) P[\min(X, Y) > 1/3] = P[XY < 1/2]$$

$$= \frac{1}{2} + \int_{1/2}^1 \int_0^{1/x} 1 \cdot dy dx = \frac{1}{2} + \int_{1/2}^1 \frac{1}{2x} dx$$

$$= \frac{1}{2} + \frac{1}{2} \ln|x| \Big|_{1/2}^1 = \underline{0.85}$$



$$\begin{aligned} \textcircled{11} \text{ (a)} \quad 1 &= \int_0^1 \int_0^1 \int_0^1 k(x+y+z) \, dx \, dy \, dz \\ &= k \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z \right) dy \, dz = k \int_0^1 \left[\left(\frac{1}{2} + z \right) + \frac{1}{2} \right] dz \\ &= k \left[1 + \frac{1}{2} \right] \Rightarrow k = \frac{2}{3} \end{aligned}$$

$$\text{(b)} \quad f_{x,y}(x,y) = \frac{2}{3} \int_0^1 (x+y+z) \, dz = \frac{2}{3} \left[x+y + \frac{1}{2} \right]$$

$$f_z(z|x,y) = \frac{f_{x,y,z}(x,y,z)}{f_{x,y}(x,y)} = \frac{x+y+z}{x+y+\frac{1}{2}}$$

$$\textcircled{12} \text{ (a)} \quad f_{x,y,z}(x,y,z) = \frac{3}{4\pi} \quad \because V = \frac{4}{3}\pi$$

$$\begin{aligned} f_x(x,y) &= \int_{-\infty}^{\infty} f_{x,y,z}(x,y,z) \, dz = \int_{-(1-x^2-y^2)^{1/2}}^{(1-x^2-y^2)^{1/2}} \frac{3}{4\pi} \, dz \\ &= \frac{3}{2\pi} (1-x^2-y^2)^{1/2} \end{aligned}$$

$$\text{(b)} \quad f_x(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y,z}(x,y,z) \, dz' \, dy' = \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} \int_{-(1-x^2-y'^2)^{1/2}}^{(1-x^2-y'^2)^{1/2}} \frac{3}{4\pi} \, dz' \, dy'$$

$$= \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} \frac{3}{2\pi} (1-x^2-y'^2)^{1/2} \, dy' \quad \text{put } a^2 = 1-x^2$$

$$\begin{aligned} \text{Consider } \int_0^a \sqrt{a^2-t^2} \, dt &= \int_0^{\pi/2} a \cos u \cdot a \cos u \, du = \frac{1}{2} a^2 \int_0^{\pi/2} (1+\cos 2u) \, du \\ &= \frac{1}{2} a^2 \left(\frac{\pi}{2} + \frac{1}{2} \sin 2u \Big|_0^{\pi/2} \right) = \frac{1}{4} \pi a^2 \end{aligned}$$

$$\therefore f_x(x) = \frac{3}{2\pi} \cdot 2 \cdot \frac{1}{4} \pi a^2 = \frac{3}{4} a^2 = \frac{3}{4} (1-x^2)$$

$$\text{(c)} \quad f(x,y|z) = \frac{f_{x,y,z}(x,y,z)}{f_z(z)} = \frac{\frac{3}{4\pi}}{\frac{3}{4}(1-z^2)} = \frac{1}{\pi(1-z^2)}$$

(d) x, y, z are not independent random variables
 $f_x(x) \cdot f_y(y) \cdot f_z(z) \neq f_{x,y,z}(x,y,z)$

$$(13) F_Z(z) = P[Z \leq z] = P[|X-Y| \leq z]$$

$$= \int_0^z \int_0^{x+z} f_{XY}(x,y) dy dx + \int_z^\infty \int_{x-z}^{x+z} f_{XY}(x,y) dy dx$$

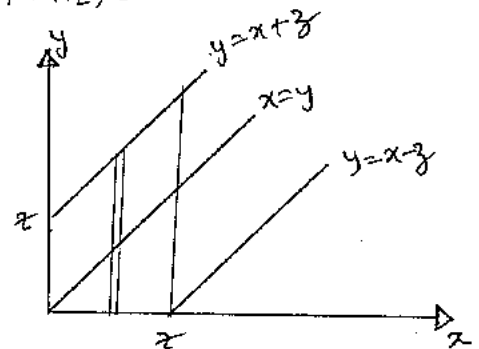
$$= \int_0^z \lambda_1 e^{-\lambda_1 x} (1 - e^{-\lambda_2(x+z)}) dx + \int_z^\infty \lambda_1 e^{-\lambda_1 x} (e^{-\lambda_2(x-z)} - e^{-\lambda_2(x+z)}) dx$$

$$= (1 - e^{-\lambda_1 z}) - \int_0^z \lambda_1 e^{-\lambda_2 z} e^{-(\lambda_1 + \lambda_2)x} dx + \int_z^\infty \lambda_1 e^{\lambda_2 z} e^{-(\lambda_1 + \lambda_2)x} dx$$

$$= (1 - e^{-\lambda_1 z}) - \frac{\lambda_1 e^{-\lambda_2 z}}{\lambda_1 + \lambda_2} + \frac{\lambda_1 e^{\lambda_2 z}}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)z}$$

$$= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 z} - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 z}$$

$$f_Z(z) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} (e^{-\lambda_2 z} + e^{-\lambda_1 z}) \quad z > 0$$

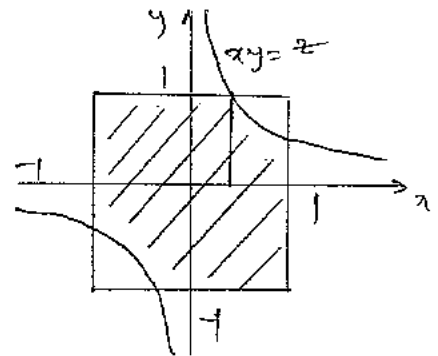


$$(14) z = XY$$

$$f_{XY}(x,y) = \frac{1}{4} \quad \begin{matrix} -1 < x < 1 \\ -1 < y < 1 \end{matrix}$$

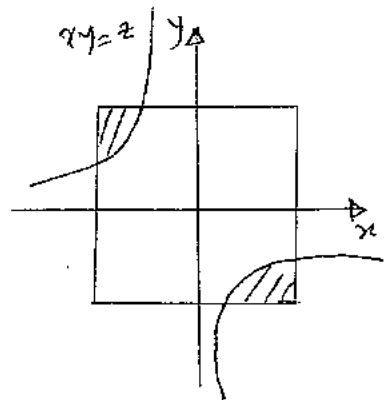
For $0 < z \leq 1$

$$\begin{aligned} P[XY \leq z] &= \frac{1}{4} \times \text{shaded area} \\ &= \frac{1}{4} \left[2 - 2(z + \int_z^1 \int_0^{z/x} dy dx) \right] \\ &= \frac{1}{2} [1 + z - z \ln z] \end{aligned}$$



For $-1 \leq z < 0$

$$\begin{aligned} P[XY \leq z] &= \frac{1}{4} \times \text{shaded area} \\ &= \frac{1}{4} \left[2 \left(1 - (-z + \int_{-z}^1 \int_{z/x}^0 dy dx) \right) \right] \\ &= \frac{1}{2} (1 + z - z \ln(-z)) \end{aligned}$$



$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} -\frac{\ln z}{2} & 0 < z < 1 \\ -\frac{\ln(-z)}{2} & -1 \leq z < 0 \end{cases}$$

(15)

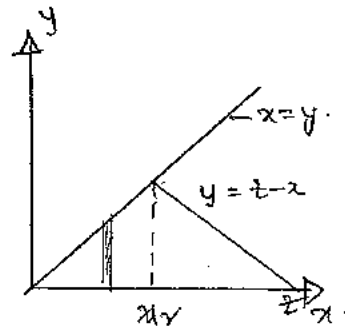
$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_0^z 2 e^{-(x+y)} dy dx + \int_{z/2}^z \int_0^{z-x} 2 e^{-(x+y)} dy dx$$

$$= (1 - 2e^{-z/2} - e^{-z}) + (2e^{-z/2} - 2e^{-z} - ze^{-z})$$

$$= 1 - e^{-z} - ze^{-z}$$

$$f_Z(z) = ze^{-z} \quad z > 0$$



(16)

$$(a) \underline{z} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_A \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad |A|=1$$

$$X_1 = U$$

$$X_2 = V - X_1 = V - U$$

$$X_3 = W - X_1 - X_2 = W - V$$

$$f_{\underline{z}}(u, v, w) = \frac{f_{\underline{X}}(\underline{x})}{|A|} \Big|_{\underline{x} = A^{-1}\underline{u}} = f_{\underline{X}}(u, v-u, w-v)$$

$$(b) f_{\underline{z}}(u, v, w) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-\frac{u^2}{2}} e^{-(v-u)^2/2} e^{-(w-v)^2/2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-\frac{1}{2}[2u^2 + 2v^2 + w^2 - 2uv - 2vw]}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-[u^2 + v^2 + \frac{1}{2}w^2 - uv - vw]}$$

(17) (a) $E[(X+Y)^2] = E[X^2 + Y^2 + 2XY] = E(X^2) + E(Y^2) + 2E(XY)$

(b) $VAR[X+Y] = E[(X+Y)^2] - E[X+Y]^2 = E(X^2) + E(Y^2) + 2E(XY) - E(X)^2 - E(Y)^2 - 2E(X)E(Y)$

$$= VAR(X) + VAR(Y) + 2[E(XY) - E(X)E(Y)]$$

$$= VAR(X) + VAR(Y) + 2COV(XY)$$

(c) $VAR[X+Y] = VAR(X) + VAR(Y)$ if $E(XY) = E(X)E(Y)$ that's uncorrelated

(18)

$$E[|x-y|] = \int_0^{\infty} \int_0^{\infty} |x-y| e^{-(x+y)} dx dy$$

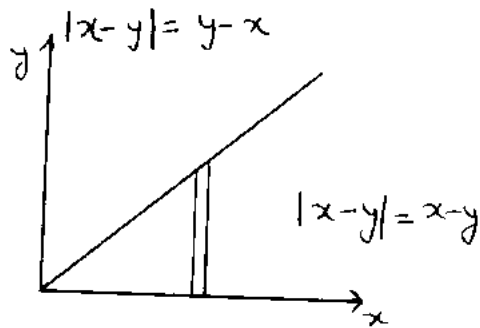
$$= 2 \int_0^{\infty} \int_0^x (x-y) e^{-x-y} dy dx$$

$$= 2 \int_0^{\infty} e^{-x} \left[x(1-e^{-x}) - \int_0^x y e^{-y} dy \right]$$

$$= 2 \int_0^{\infty} (x e^{-x} + e^{-2x} - e^{-x}) dx$$

$$= 2 \left[1 + \frac{1}{2} - 1 \right]$$

$$= 1$$



$$(19) E[X^2 Y] = E(X^2) E(Y) \quad \text{since } X, Y \text{ are independent.}$$

$$= 1 \cdot 1 = \underline{\underline{1}}$$

$$\text{since } \text{Var}(X) = E(X^2) - E(X)^2 \\ 1 = E(X^2) - 0 \Rightarrow E(X^2) = 1$$

$$E(Y) = \frac{b+9}{2} = \frac{3-1}{2} = 1$$

$$(20) (a) 1 = \int_0^{\pi/2} \int_0^{\pi/2} c \sin(x+y) dx dy \\ = c \int_0^{\pi/2} [-\cos(x+y)]_0^{\pi/2} dy$$

$$= c \int_0^{\pi/2} [\cos y - \cos(\frac{\pi}{2} + y)] dy = c \int_0^{\pi/2} (\cos y + \sin y) dy$$

$$= 2c \sin y \Big|_0^{\pi/2} = 2c \Rightarrow c = \frac{1}{2}$$

$$(b) F_{XY}(x, y) = \int_0^y \int_0^x \frac{1}{2} \sin(u+v) du dv = \int_0^y \left[-\frac{1}{2} \cos(u+v) \right]_0^x dv$$

$$= \frac{1}{2} \int_0^y (\cos v - \cos(x+v)) dv = \frac{1}{2} (\sin v - \sin(x+v)) \Big|_0^y$$

$$= \frac{1}{2} (\sin y - \sin(x+y) + \sin x)$$

$$(c) f_X(x) = \int_0^{\pi/2} \frac{1}{2} \sin(x+y') dy' = \frac{1}{2} (-\cos(x+y')) \Big|_0^{\pi/2}$$

$$= \frac{1}{2} (\cos x + \sin x)$$

$$f_Y(y) = \frac{1}{2} (\cos y + \sin y)$$

$$\begin{aligned}
 (d) \quad E(X) &= \int_0^{\pi/2} x \frac{1}{2} (\cos x + \sin x) dx \\
 &= \frac{1}{2} \int_0^{\pi/2} x \cos x dx - \frac{1}{2} \int_0^{\pi/2} x \sin x dx \\
 &= \frac{1}{2} \left[x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \right] - \frac{1}{2} \left[x \cos x \Big|_0^{\pi/2} - \int_0^{\pi/2} \cos x dx \right] \\
 &= \frac{1}{2} \left[\frac{\pi}{2} - 1 \right] + \frac{1}{2} = \frac{\pi}{4}
 \end{aligned}$$

$$E(Y) = \frac{\pi}{4}$$

$$\begin{aligned}
 E(X^2) &= \int_0^{\pi/2} x^2 \frac{1}{2} (\cos x + \sin x) dx \\
 &= \frac{1}{2} \int_0^{\pi/2} x^2 \cos x dx - \frac{1}{2} \int_0^{\pi/2} x^2 \sin x dx = \frac{1}{2} x^2 \sin x \Big|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin 2x dx \\
 &\quad - \frac{1}{2} x^2 \cos x \Big|_0^{\pi/2} + \frac{1}{2} \int_0^{\pi/2} \cos 2x dx
 \end{aligned}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} \right)^2 - 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \left(\frac{\pi}{2} - 1 \right) = \frac{\pi^2}{8} + \frac{\pi}{2}$$

$$\text{VAR}(X) = E(X^2) - [E(X)]^2 = \frac{\pi^2}{16} + \frac{\pi}{2}$$

$$\text{VAR}(Y) = \frac{\pi^2}{16} + \frac{\pi}{2}$$

$$\begin{aligned}
 E(XY) &= \iint xy \frac{1}{2} \sin(x+y) dx dy = \frac{1}{2} \int_0^{\pi/2} y dy \int_0^{\pi/2} -x \cos(x+y) dx \\
 &= -\frac{1}{2} \int_0^{\pi/2} y \left[x \cos(x+y) \Big|_0^{\pi/2} - \int_0^{\pi/2} \cos(x+y) dx \right] dy \\
 &= -\frac{1}{2} \int_0^{\pi/2} y \cdot \left[-\frac{\pi}{2} \sin y - \sin(x+y) \Big|_0^{\pi/2} \right] dy \\
 &= \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{2} y \sin y dy + \frac{1}{2} \int_0^{\pi/2} y (\cos y - \sin y) dy \\
 &= \left(\frac{\pi}{2} - 1 \right) \int_0^{\pi/2} \frac{1}{2} y \sin y dy + \int_0^{\pi/2} \frac{1}{2} y \cos y dy \\
 &= \pi/2 - 1
 \end{aligned}$$

$$\therefore \text{COV}(X, Y) = E(XY) - E(X)E(Y) = \frac{\pi}{2} - 1 - \left(\frac{\pi}{4} \right)^2$$