# EE 3140 Homework #2 solutions

① (a) The sample space has 100 elements, each element corresponding to a bill.

$$S = \{ \alpha_1, \alpha_2, \alpha_3 \cdots\cdots \alpha_{100} \}$$

where $\alpha_i$ represents the $i^{th}$ bill.

Since all bills are equiprobable $P[\alpha_i] = \frac{1}{100}$

(b) $S_X = \{1, 5, 50\}$

$P[X=1] = \frac{90}{100}$

$P[X=5] = \frac{9}{100}$

$P[X=50] = \frac{1}{100}$

② $S_Y = \{1, 2, 3, 4\}$

$P[Y=1] = P(a) = \frac{1}{2}$

$P[Y=2] = P(b) = \frac{1}{4}$

$P[Y=3] = P(c) = \frac{1}{8}$

$P[Y=4] = p(d) + p(e) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

③

$P[U>0] = 1 - P[U \leq 0]$

$= 1 - F_U(0) = \frac{1}{2}$

$P[U<5] = 1$

$P[|U| < \frac{1}{3}] = F_U(\frac{1}{3}) - F_U(-\frac{1}{3})$

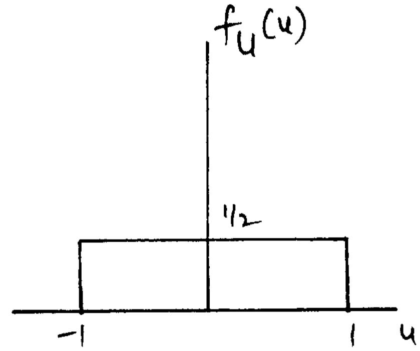
$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

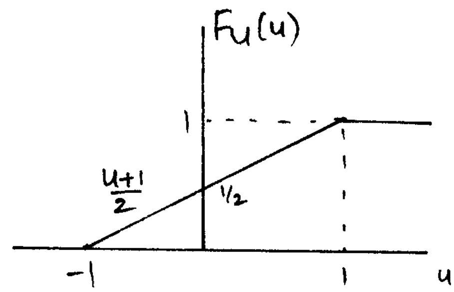
$P[|U| \geq \frac{3}{4}] = 1 - P[|U| < \frac{3}{4}]$

$= 1 - F_U(\frac{3}{4}) + F_U(-\frac{3}{4})$

$= \frac{1}{4}$

$P[\frac{1}{3} < U < \frac{1}{2}] = F_U(\frac{1}{2}) - F_U(\frac{1}{3}) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$




④ $P(A) = P(X > 1/3) = 1 - P(X \leq \frac{1}{3}) = 1 - F_X(\frac{1}{3}) = 0$

$P(B) = P[|X| \geq 1] = 1 - P[|X| < 1] = 1 - F_X(1) + F_X(-1) = \frac{1}{3}$

$P(C) = P[|X - \frac{1}{3}| < 1] = P[-\frac{2}{3} < X < \frac{4}{3}] = F_X(\frac{4}{3}) - F_X(-\frac{2}{3}) = \frac{16}{27}$

$P(D) = P(X<0) = F_X(0) = 1$

⑤ (a) X is a mixed type of random variable.

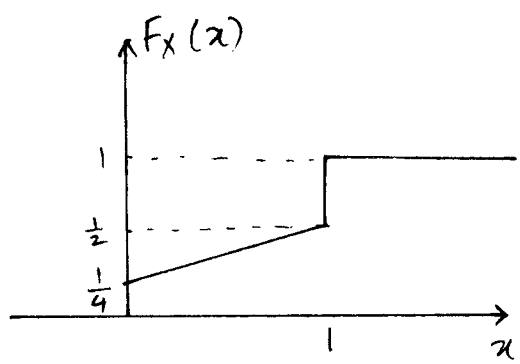
(b) $P(X < -\frac{1}{2}) = 0$

$P(X<0) = 0$

$P(X \leq 0) = \frac{1}{4}$

$P(\frac{1}{4} \leq X \leq 1) = F_X(1^-) - F_X(\frac{1}{4})$

$= \frac{1}{2} - \frac{5}{16} = \frac{3}{16}$



$P(\frac{1}{4} \leq X \leq 1) = F_X(1^+) - F_X(\frac{1}{4}) = 1 - \frac{5}{16} = \frac{11}{16}$

$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F_X[\frac{1}{2}] = \frac{5}{8}$

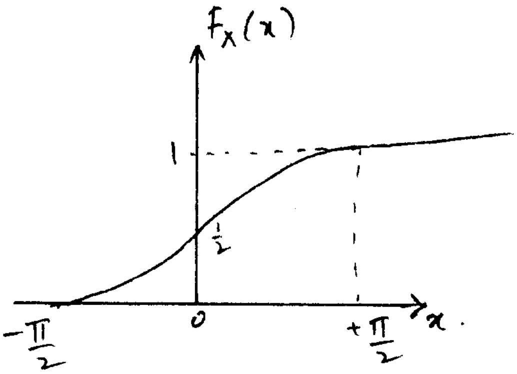
$P(X \geq 5) = 1 - P(X < 5) = 1 - 1 = 0$

$P(X < 5) = 1$

⑥ Since $X$ is continuous $F_X(\frac{\pi}{2}) = 1$

$\Rightarrow c(1 + \sin(\frac{\pi}{2})) = 1$

$\Rightarrow c = \frac{1}{2}$




$\therefore F_X(x) = 0 \qquad x \leq -\frac{\pi}{2}$

$= \frac{1}{2}(1 + \sin x) \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$= 1 \qquad \frac{\pi}{2} \leq x$

⑦ $P[\sigma \leq R \leq 2\sigma] = F_R(2\sigma) - F_R(\sigma)$

$= 1 - e^{-\frac{4\sigma^2}{2\sigma^2}} - 1 + e^{-\frac{\sigma^2}{2\sigma^2}}$

$= \underline{e^{-\frac{1}{2}} - e^{-2}}$

$P[R > 3\sigma] = 1 - P[R \leq 3\sigma] = 1 - F_R(3\sigma)$

$= 1 - 1 + e^{-\frac{9\sigma^2}{2\sigma^2}}$

$= \underline{e^{-9/2}}$

⑧ (a) Since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\Rightarrow \int_{-1}^{0} c\,x(1-x)\,dx = c\int_{-1}^{0} x - x^2\,dx = 1$$

$$\Rightarrow c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0} = c\left[-\frac{5}{6}\right] = 1 \Rightarrow \underline{c = -\frac{6}{5}}$$

(b) $P\left[\frac{1}{2} \leqslant X \leqslant \frac{3}{4}\right] = \int_{\frac{1}{2}}^{3/4} f_X(x)\,dx = 0$

$\because f_X(x) = 0$ in this range.

(c) $F_X(x) = \int_{-1}^{x} -\frac{6}{5}\cdot(x - x^2)\,dx$

$$= -\frac{6}{5}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{x} = 1 - \frac{3}{5}x^2 + \frac{2}{5}x^3$$

$$\therefore F_X(x) = \begin{cases} 0 & x \leqslant -1 \\ 1 - \frac{3}{5}x^2 + \frac{2}{5}x^3 & -1 \leqslant x \leqslant 0 \\ 1 & 0 < x \end{cases}$$

⑨ Since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\Rightarrow \int_{-1}^{1} c(1 - x^4) = 1 \Rightarrow c\left[x - \frac{x^5}{5}\right]_{-1}^{1} = 1$$

$$\Rightarrow \underline{c = \frac{5}{8}}$$

(b) $F_X(x) = \int_{-1}^{x} f_X(x')\,dx'$ for $-1 \leq x \leq 1$

$$= \int_{-1}^{1} \frac{5}{8}(1 - x'^4)\,dx' = \frac{5}{8}\left[x' - \frac{x'^5}{5}\right]_{-1}^{x} = \underline{\frac{1}{2} + \frac{5}{8}x - \frac{1}{8}x^5}$$

$\therefore F_X(x) = 0 \quad ; x < -1$

$= \frac{1}{2} + \frac{5}{8}x - \frac{1}{8}x^5 \quad ; -1 \leq x \leq 1$

$= 1 \quad ; x > 1$

(c) $P(|X| < \frac{1}{2})$

$= F_X(\frac{1}{2}) - F_X(-\frac{1}{2}) =$

$= \frac{1}{2} + \frac{1}{8}\cdot\frac{5}{2} - \frac{1}{8}\left(\frac{1}{2}\right)^5 - \frac{1}{2} + \frac{1}{8}\left(\frac{-5}{2}\right) - \frac{1}{8}\left(\frac{1}{2}\right)^5$

$= \underline{\underline{\frac{79}{128}}}$

10 (a) $f_X(x) = 0 \quad ; |x| > a$

$= c\left(1 - \frac{|x|}{a}\right) \quad ; |x| \leq a$

$\because$ area under the curve $= 1$

$\frac{1}{2}\cdot 2a \cdot c = 1 \Rightarrow \underline{\underline{c = \frac{1}{a}}}$

(b) $F_X(x) = 0 \quad x < -a$

for $-a \leq x \leq a$

$F_X(x) = \cancel{\frac{1}{a}} \int_{-\infty}^{x} f_X(x)\,dx = \frac{1}{a}\int_{-a}^{x} 1 + \frac{x}{a}\,dx = \frac{1}{2} + \frac{1}{a}\left[x + \frac{x^2}{2a}\right]$

For $0 \leq x \leq a$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-a}^{0} f_X(x)\,dx + \int_{0}^{x} f_X(x)\,dx$$

$$= \frac{1}{2} + \frac{1}{a}\int_{0}^{x} 1 - \frac{x}{a}\,dx$$

$$= \frac{1}{2} + \frac{1}{a}\left[x - \frac{x^2}{2a}\right]$$

For $x > a$;

$$F_X(x) = 1$$

(c) $P[\,|X| < b\,] = \frac{1}{2}$

$$\Rightarrow \int_{-b}^{b} f_X(x)\,dx = \frac{1}{2} \Rightarrow 2\int_{0}^{b} \frac{1}{a} - \frac{x}{a^2}\,dx = \frac{1}{2}$$

$$\Rightarrow 2\left[\frac{x}{a} - \frac{x^2}{2a^2}\right]_0^b = \frac{1}{2} \Rightarrow 2\left[\frac{b}{a} - \frac{b^2}{2a^2}\right] = \frac{1}{2}$$

Solving for $b$

we get $b = a\left(1 - \frac{1}{\sqrt{2}}\right)$.

(11) Given $Y = X^2 \Rightarrow X = -\sqrt{Y}, +\sqrt{Y}$

$$\left|\frac{dY}{dX}\right| = |2X| = 2\sqrt{Y}$$

$$\therefore F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq +\sqrt{y})$$

$$\Rightarrow F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$ //

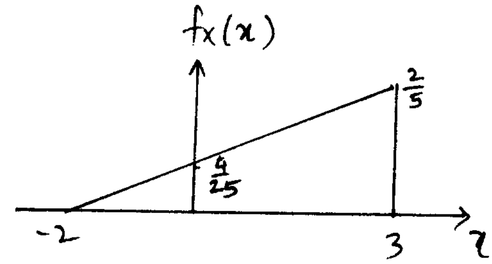
Differentiating w.r.t. $y$.

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$ //

(12) Since total area under $f_X(x) = 1$

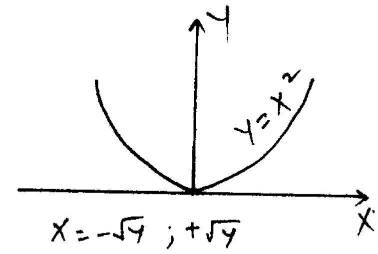
$$\Rightarrow \frac{1}{2} \cdot 5 \cdot h = 1$$

$$h = \frac{2}{5}$$




$$\text{slope } m = \frac{\frac{2}{5}}{5} = \frac{2}{25}$$

$$c = 4/25$$




$$\therefore f_X(x) = \frac{2}{25}x + \frac{4}{25} \quad ; \quad -2 \leq x \leq 3$$

For $-ve$ $x$

$$f_Y(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} = \left[-\frac{2}{25}\sqrt{y} + \frac{4}{25}\right]\frac{1}{2\sqrt{y}} \quad ; \quad 0 < y \leq 4$$

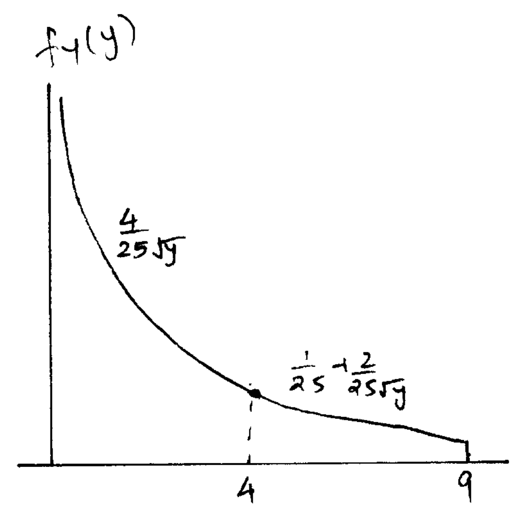
For $+ve$ $x$

$$f_Y(y) = \frac{f_X(+\sqrt{y})}{2\sqrt{y}} = \left[\frac{2}{25}\sqrt{y} + \frac{4}{25}\right]\frac{1}{2\sqrt{y}} \quad 0 < y \leq 9$$

Combining,

$$f_Y(y) = \left[-\frac{2}{25}\sqrt{y} + \frac{4}{25}\right]\frac{1}{2\sqrt{y}} + \left[\frac{2}{25}\sqrt{y} + \frac{4}{25}\right]\frac{1}{2\sqrt{y}}$$

$$= \frac{4}{25\sqrt{y}} \quad 0 \leq y \leq 4$$



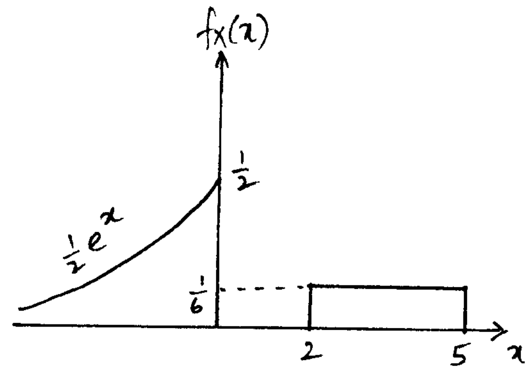

$$f_Y(y) = \frac{1}{25} + \frac{2}{25\sqrt{y}} \quad 4 \leq y \leq 9$$

(13) $f_X(x) = \frac{1}{2} e^{x} \qquad x \leq 0$

$= 0 \qquad 0 \leq x \leq 2$

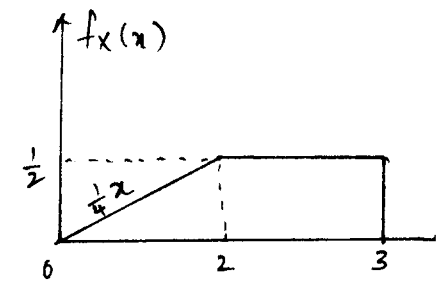
$= \frac{1}{6} \qquad 2 \leq x \leq 5$

$= 1 \qquad x \geq 5$




(14) Since total area under $f_X(x) = 1$

$\Rightarrow \quad \frac{1}{2} \cdot 2 \cdot h + 1 \cdot h = 1 \quad \Rightarrow h = \frac{1}{2}$




$$E(X) = \int_0^2 x f_X(x)\, dx + \int_2^3 x f_X(x)\, dx$$

$$= \int_0^2 \frac{1}{4} x^2 dx + \int_2^3 \frac{1}{2} x\, dx = \frac{1}{4} \frac{x^3}{3}\Big|_0^2 + \frac{1}{2} \frac{x^2}{2}\Big|_2^3$$

$$= \frac{23}{12}$$

$$E(X^2) = \int_0^2 x^2 f_X(x)\, dx + \int_2^3 x^2 f_X(x)\, dx$$

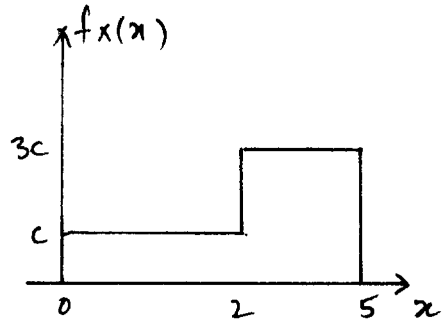
$$= \int_0^2 \frac{1}{4} x^3 dx + \int_2^3 \frac{1}{2} x^2 dx = \frac{1}{4} \frac{x^4}{4}\Big|_0^2 + \frac{1}{2} \frac{x^3}{3}\Big|_2^3$$

$$= \frac{25}{6}$$

$$Var(x) = E(X^2) - E(X)^2 = \frac{25}{6} - \left(\frac{23}{12}\right)^2 = 0.493$$

(15) $2c + 3.3c = 1$

$\Rightarrow c = \frac{1}{11}$




$$E(x) = \int_0^2 x \cdot \frac{1}{11} dx + \int_2^3 x \cdot \frac{3}{11} dx$$

$$= \frac{1}{11} \left. \frac{x^2}{2} \right|_0^2 + \frac{3}{11} \cdot \left. \frac{x^2}{2} \right|_2^3 = \frac{67}{22}$$

$$E(x^2) = \int_0^2 x^2 \cdot \frac{1}{11} dx + \int_2^3 x^2 \cdot \frac{3}{11} dx$$

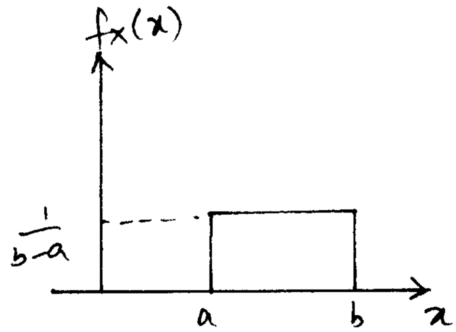
$$= \frac{1}{11} \left. \frac{x^3}{3} \right|_0^2 + \frac{3}{11} \left. \frac{x^3}{3} \right|_2^3 = \frac{359}{33}$$

$$Var(x) = E(x^2) - E(x)^2 = \frac{359}{33} - \left(\frac{67}{22}\right)^2 = 1.604$$

(16) $E(X) = \frac{2+1+1+3+4+2+0+1+2+0+7+1}{12} = 2$

$E(x^2) = \frac{4+1+1+9+16+4+0+1+4+0+49+1}{12} = 7.5$

$Var(X) = 7.5 - 4 = 3.5$




(17) $$E(x) = \int_a^b x \cdot \frac{1}{(b-a)} dx$$

$$= \frac{1}{(b-a)} \cdot \frac{b^2-a^2}{2} = \frac{a+b}{2}$$

$$E(x^2) = \frac{1}{b-a} \cdot \int_a^b x^2 dx = \frac{1}{b-a} \frac{(b^3-a^3)}{3} = \frac{b^2+ab+a^2}{3}$$

$$Var(x) = \frac{b^2+ab+a^2}{3} - \left[\frac{a+b}{2}\right]^2 = \frac{(b-a)^2}{12}$$

(18) $E(X) = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1} = p \sum_{k=1}^{\infty} k(1-p)^{k-1}$

but $\int k(1-p)^{k-1} dp = (1-p)^k$

and $\sum_{k=1}^{\infty} (1-p)^k = \frac{1}{p}$

$\frac{d}{dp} \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}$

$\therefore E(X) = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{p}{p^2} = \frac{1}{p}$

$E(X^2) = \sum_{k=1}^{\infty} k^2(1-p)^{k-1} = p \sum_{k=1}^{\infty} k^2(1-p)^{k-1}$

$\therefore \int k^2(1-p)^{k-1} = k(1-p)^k = (1-p)\, k(1-p)^{k-1}$

$\therefore \sum_{k=1}^{\infty} (1-p)\, k\, (1-p)^{k-1} = (1-p) \cdot \frac{1}{p^2}$

$\frac{d}{dp} \int k^2(1-p)^{k-1} dp = \frac{(2-p)}{p^3}$

$\therefore Var(X) = E(X^2) - E(X)^2$

$= \frac{1-p}{p^2}$

(19) $$E(X) = \sum_{k=1}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{\infty} k \frac{n!}{k!(n-k)!} p^k \cdot (1-p)^{n-k}$$

$$= np \cdot \sum_{k=1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k)!} \cdot p^{k-1}(1-p)^{n-k}$$

$$= \underline{np}$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{\infty} \frac{k(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}.$$

$$= np \sum_{k=1}^{\infty} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} \cdot p^{k-1} (1-p)^{n-k}$$

$$= np \left[ \sum_{k=1}^{\infty} \frac{(k-1)(n-1)!}{(k-1)!(n-k)!} \cdot p \cdot p^{k-2} (1-p)^{n-k} + \sum_{k=1}^{\infty} \frac{(1)(n-1)!}{(k-1)!(n-k)!} \cdot p^{k-1} (1-p)^{n-k} \right]$$

$$= np \left[ (n-1)p \sum_{k=1}^{\infty} \frac{(n-2)!}{(k-2)!(n-k)} p^{k-2} (1-p)^{n-k} + \sum_{k=1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} \cdot (1-p)^{n-k} \right]$$

$$= np\left[ (n-1)p + 1 \right] = n^2p^2 - np^2 + np$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \underline{np(1-p)}$$

(20) $E(X) = \sum_{K=0}^{\infty} K \cdot \frac{\alpha^K e^{-\alpha}}{K!}$

$= e^{-\alpha}\left[\alpha + \frac{2\alpha^2}{2!} + \frac{3\alpha^3}{3!} + \cdots\right]$

$= \alpha e^{-\alpha}\left[1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots\right]$

$= \alpha e^{-\alpha} e^{+\alpha} = \underline{\underline{\alpha}}$

$E(X^2) = \sum_{K=0}^{\infty} K^2 \cdot \frac{\alpha^K e^{-\alpha}}{K!}$

$= \alpha \sum_{K=0}^{\infty} \frac{K \alpha^{K-1}}{(K-1)!} e^{-\alpha} = \alpha \sum_{K-1=0}^{\infty} (K-1+1) \frac{\alpha^{K-1}}{(K-1)!} e^{-\alpha}$

$= \alpha \left[ \underbrace{\sum_{K-1=0}^{\infty} \frac{(K-1)\alpha^{K-1} e^{-\alpha}}{(K-1)!}}_{\alpha} + \underbrace{\sum_{K-1=0}^{\infty} \frac{\alpha^{K-1} e^{-\alpha}}{(K-1)!}}_{1} \right]$

$= \alpha[\alpha + 1] \quad = \alpha^2 + \alpha$

$Var(X) = E(X^2) - E(X)^2$

$= \alpha^2 + \alpha - \alpha^2 \quad \underline{\underline{= \alpha}}$