Homework #3 Due: next week

1. An urn contains one black ball and three white balls. Four balls are drawn from the urn. Let $I_k=1$ if the outcome of the kth draw is the black ball and let $I_k=0$ otherwise. Define the following three random variables:

   $X = I_1 + I_2 + I_3 + I_4$
   $Y = \min\{ I_1, I_2, I_3, I_4 \}$, and
   $Z = \max\{ I_1, I_2, I_3, I_4 \}$.

   a. Find the probability law for (X, Y, Z) if each ball is put back into the urn after each draw.
   b. Find the probability law for (X, Y, Z) if each ball is not put back into the urn after each draw.

2. Let the random variables X, Y, and Z be independent random variables. Find the following probabilities in terms of $F_X(x)$, $F_Y(y)$ and $F_Z(z)$.
   a. $P[|X| < 5, Y > 2, Z^2 >= 2]$
   b. $P[X > 5, Y < 0, Z = 1]$
   c. $P[\min(X, Y, Z) > 2]$
   d. $P[\max(X, Y, Z) < 6]$

3. A die is tossed twice; let $X1$ and $X2$ denote the outcome of the first and second toss, respectively.
   a. What is the joint pmf for $(X1, X2)$ if the tosses are independent and if the outcomes of each toss are equiprobable?
   b. Let $X = \min(X1, X2)$ and $Y = \max(X1, X2)$. Find the joint pmf for (X, Y).
   c. Find the marginal pmf's for X and Y in part b.

4. a. Find the marginal pmf's for the pairs of random variables with the indicated joint pmf.

i.	*X*		
Y	-1	0	1
-1	1/6	0	1/6
0	0	1/3	0
1	1/6	0	1/6

ii.	*X*		
Y	-1	0	1
-1	1/9	1/9	1/9
0	1/9	1/9	1/9
1	1/9	1/9	1/9

iii.	*X*		
Y	-1	0	1
-1	0	0	1/3
0	0	1/3	0
1	1/3	0	0

   c. Find the probability of the event $A = \{X <= 0\}$, $B = \{X <= Y\}$, and $C = \{X = -Y\}$ for the above joint pmf's.

5. Let X and Y denote the amplitude of noise signals at two antennas. The random vector (X, Y) has the joint pdf

   $f_{X,Y}(x, y) = axe^{-ax2/2}bye^{-by2/2}$ $\qquad x>0, y>0, a>0, b>0$

   a. Find the joint cdf
   b. Find $P[X > Y]$
   c. Find the marginal pdf's

6. The random vector variable (X, Y) has the joint pdf

   $f_{X,Y}(x, y) = k(x + y)$ $\qquad 0<x<1, 0<y<1$

   a. Find k.
   b. Find the joint cdf of (X, Y).
   c. Find the marginal pdf of X and of Y.

7. The random vector (X, Y) has a joint pdf

   $f_{X,Y}(x, y) = 2xe^{-x}ye^{-2y}$ $\qquad x>0, y>0$

   Find the probability of the following events:
   a. $\{X + Y <= 8\}$.
   b. $\{X < Y\}$.
   c. $\{X - Y <= 10\}$.
   d. $\{X^2 < Y\}$.

8. Let (X, Y) have the joint pdf

$f_{X,Y}(x, y) = xe^{-x(1+y)}$ $\quad x>0, y>0$

Find the marginal pdf of X and of Y.

9. Let X and Y be independent random variables that are uniformly distributed in [0, 1]. Find the probability of the following events:
   a. $\{a < X < b\} \cap \{Y <= d\}$
   b. $\{a <= X <= b\} \cap \{c <= Y <= d\}$
   c. $\{|X| > a\} \cap \{c <= Y <= d\}$

10. Let X and Y be independent random variables. Find an expression for the probability of the following events in terms of $F_X(x)$ and $F_Y(y)$.
    a. $P[X^2 < 1/2, |Y-1| < 1/2]$
    b. $P[X/2 < 1, Y > 0]$
    c. $P[XY < 1/2]$
    d. $P[\min(X, Y) > 1/3]$

11. Let X, Y, Z have joint pdf

    $f(x, y, z) = k(x + y+z)$ $\quad 0<= x <=1, 0<= y<=1, 0<= z <= 1$

    a. Find k.
    b. Find $f_Z(z\,|x, y)$

12. A point (X, Y, Z) is selected at random inside the unit sphere.
    a. Find the marginal joint pdf of X and Y.
    b. Find the marginal pdf of X.
    c. Find the conditional joint pdf of X and Y given Z.
    d. Are X, Y, and Z independent random variables?

13. Let X and Y be independent exponential random variables. Find the pdf of $Z = |X- Y|$.

14. Let X and Y be independent random variables that are uniformly distributed in the interval [-1, 1]. Find the pdf of $Z = XY$.

15. The random variables X and Y have the joint pdf

    $f_{X,Y}(x, y) = 2e^{-(x+y)}$ $\quad 0<= y <= x < \infty$

    Find the pdf of $Z = X + Y$. *Note*: X and Y are not independent.

16. a. Find the joint pdf of

    $U = X_1,$

    $U = X_1+ X_2,$

    $U = X_1+ X_2+ X_3.$

    c. Evaluate the joint pdf of (U, V, W) if the x_i are independent zero-mean, unit-variance Gaussian random variables.

17. a. Find $E[(X + Y)^2]$.
    b. Find the variance of $X + Y$.
    d. Under what condition is the variance of the sum equal to the sum of the individual variance?

18. Find $E[|X- Y|]$ if X and Y are independent exponential random variables with parameter $\alpha = 1$.

'9. Find $E[X^2Y]$ where X is a zero-mean, unit-variance Gaussian random variable, and Y is a uniform random variable in the interval [-1, 3], and X and Y are independent.

0. The random variables X and Y have joint pdf

   $f_{X,Y}(x, y) = c \sin(x+y)$ $\quad 0<= x < \pi/2, 0<= y < \pi/2$

   a. Find the value of constant c.
   b. Find the joint cdf of (X, Y).
   c. Find the marginal joint pdf of X and Y.
   d. Find the mean, variance, and covariance of X and Y.