EE 2720

Handout #8

• Duality

In handout #5 we presented the ten axioms of switching algebra. We present these axioms here again. These axioms are:

(A1) $X=0$ if $X\neq 1$	(A1′) $X=1$ if $X\neq 0$.
(A2) If $X=0$, then $X'=1$	(A2′) If $X=1$ then $X'=0$
(A3) $0\cdot 0=0$	(A3′) $1+1=1$
(A4) $1\cdot 1=1$	(A4′) $0+0=0$
(A5) $0\cdot 1=1\cdot 0=0$	(A5′) $1+0=0+1=1$

As seen from the above, these axioms are stated in pairs; ((A1)-(A1′), (A2)-(A2′), (A3)-(A3′), (A4)-(A4′), (A5)-(A5′)). The primed version of each axiom is obtained from the unprimed version by simply swapping 0 and 1 and, if present, $\cdot$ and $+$. Since the above ten axioms can be used to prove all the theorems of switching algebra, we can now state the following theorem about theorems:

• Principle of Duality: Any theorem or identity in switching algebra remains true if 0 and 1 are swapped and $\cdot$ and $+$ are swapped as well.

• Implication: Duality is important. It halves the amount that you have to learn. Once you know a switching algebra theorem, you automatically know its dual. The dual theorems are the primed versions of the unprimed.

• Important Note: Before taking the dual of a logic expression fully parenthesize it. If you do not do this you

will produce mistakes.
I will demonstrate the above with an example. It is going to be theorem (T10). I will try to take its dual without using parentheses. I will produce a mistake.

$X \cdot Y + X \cdot Y' = X$ (theorem (T10))
$X + Y \cdot X + Y' = X$ (after applying the principle of duality)
$X \cdot Y + X + Y' = X$
$X \cdot Y + X \cdot 1 + Y' = X$
$X \cdot (Y + 1) + Y' = X$
$X \cdot 1 + Y' = X$
$X + Y' = X$ which is wrong.

The correct way of doing it follows
$X \cdot Y + X \cdot Y' = X$ (theorem (T10))
$(X \cdot Y) + (X \cdot Y') = X$ (after putting parentheses).
$(X + Y) \cdot (X + Y') = X$ (after applying principle of duality).

The obtained last line $(X+Y) \cdot (X+Y') = X$ is the correct theorem (T10′) which is the dual of theorem (T10).

• <u>Dual of a logic expression</u>: If $F(X_1, X_2, \ldots, X_n, +, \cdot, ')$ is a fully parenthesized logic expression involving the variables $X_1, X_2, \ldots, X_n$ and the operators $+$, $\cdot$, and $'$, then the dual of F, denoted by F^D, is the same expression with $+$ and $\cdot$ swapped. In other words

$$F^D(X_1, X_2, \ldots, X_n, +, \cdot, ') = F(X_1, X_2, \ldots, X_n, \cdot, +, ')$$

<u>Note</u>: The generalized DeMorgan's theorem (T14) can

also be stated as follows:

$$[F(X_1, X_2, \cdots, X_n)]' = F^D(X_1', X_2', \cdots, X_n')$$

- **Standard Representations of Logic Functions**

The most basic representation of a logic function is the truth table. I already provided the definition of the truth table in handout # 5; (see page 7 of handout # 5). Below I provide the truth tables for a NOT gate (inverter), a 2-input AND gate and a 2-input OR gate. You already know this information.

A	A'
0	1
1	0

A	B	A•B
0	0	0
0	1	0
1	0	0
1	1	1

A	B	A+B
0	0	0
0	1	1
1	0	1
1	1	1

Another truth table is shown below in Table 1. This truth table is for another 3-variable logic function F(X, Y, Z).

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Table 1

Note: The information contained in a truth table can be delivered algebraically but some definitions are needed first.

- Literal: The definition of a literal has been provided in handout # 6. As a reminder, a literal is a variable or the complement of a variable.
- Product term: The definition of a product term has been provided in handout # 6. As a reminder, a product term is a single literal or a logical product of two or more literals.
- Sum-of-products expression: The definition of a sum-of-products expression has been provided in handout # 6. As a reminder, a sum-of-products expressions is a logical sum of product terms.
- Sum term: The definition of a sum term has been provided in handout # 6. As a reminder, a sum term is a single literal or a logical sum of two or more literals.
- Product-of-sums expression: The definition of a product-of-sums expression has been provided in handout # 6. As a reminder, a product-of-sums expression is a logical product of sum terms.
- Normal term: A normal term is a product or sum term in which no variable appears more than once. A nonnormal term can always be simplified to a constant or a normal term using one of the theorems (T3), (T3'), (T5), (T5'). (T3) states $X+X=X$, (T3') states $X \cdot X = X$, (T5) states $X+X'=1$, (T5') states $X \cdot X'=0$. Examples of normal terms are: $X' \cdot Y \cdot Z$, $X+Y'+Z'$. Examples of nonnormal terms are: $X' \cdot Y \cdot Y \cdot Z$, $X+X+Y'+Z'$, $X \cdot X' \cdot Y$, $X+X'+Y$, $Z \cdot Z'$, $W+W'$.

- n-variable minterm: An n-variable minterm is a normal product term with n literals. There are 2^n such product terms. Examples of 3-variable minterms are: $X' \cdot Y \cdot Z$, $X \cdot Y' \cdot Z'$, $X \cdot Y \cdot Z$. Examples of 4-variable minterms are $X \cdot Y' \cdot Z' \cdot W$, $X \cdot Y \cdot Z \cdot W$, $X \cdot Y' \cdot Z \cdot W'$
- n-variable maxterm: An n-variable maxterm is a normal sum term with n literals. There are 2^n such sum terms. Examples of 3-variable maxterms are: $X+Y'+Z$, $X'+Y+Z'$, $X+Y+Z$. Examples of 4-variable maxterms are: $X'+Y+Z+W'$, $X'+Y+Z'+W$, $X'+Y'+Z'+W'$.

Note: There is a correspondence between the truth table, minterms and maxterms. A minterm is a product term that is 1 in exactly one row of the truth table. A maxterm is a sum term that is 0 in exactly one row of the truth table. Table 2 below explains the situation for a 3-variable truth table

Row	X	Y	Z	F	Minterm	Maxterm
0	0	0	0	F(0,0,0)	$X' \cdot Y' \cdot Z'$	$X+Y+Z$
1	0	0	1	F(0,0,1)	$X' \cdot Y' \cdot Z$	$X+Y+Z'$
2	0	1	0	F(0,1,0)	$X' \cdot Y \cdot Z'$	$X+Y'+Z$
3	0	1	1	F(0,1,1)	$X' \cdot Y \cdot Z$	$X+Y'+Z'$
4	1	0	0	F(1,0,0)	$X \cdot Y' \cdot Z'$	$X'+Y+Z$
5	1	0	1	F(1,0,1)	$X \cdot Y' \cdot Z$	$X'+Y+Z'$
6	1	1	0	F(1,1,0)	$X \cdot Y \cdot Z'$	$X'+Y'+Z$
7	1	1	1	F(1,1,1)	$X \cdot Y \cdot Z$	$X'+Y'+Z'$

Table 2

Explanations: Regarding the table 2 of the previous page, the minterm $X' \cdot Y' \cdot Z'$ is 1 only in row 0 of the truth table. In row 0 of the truth table $X=0$, $Y=0$, $Z=0$, so $X'=1$, $Y'=1$, $Z'=1$ and obviously $X' \cdot Y' \cdot Z' = 1 \cdot 1 \cdot 1 = 1$. The minterm $X' \cdot Y' \cdot Z$ is 1 only in row 1 of the truth table. In row 1 of the truth table $X=0$, $Y=0$, $Z=1$, so $X'=1$, $Y'=1$ and $X' \cdot Y' \cdot Z = 1 \cdot 1 \cdot 1 = 1$. The minterm $X' \cdot Y \cdot Z'$ is 1 only in row 2 of the truth table. In row 2 of the truth table $X=0$, $Y=1$, $Z=0$, so $X'=1$, $Z'=1$ and $X' \cdot Y \cdot Z' = 1 \cdot 1 \cdot 1 = 1$. Similarly, the minterm $X' \cdot Y \cdot Z$ is 1 only in row 3 of the truth table, the minterm $X \cdot Y' \cdot Z'$ is 1 only in row 4 of the truth table, the minterm $X \cdot Y' \cdot Z$ is 1 only in row 5 of the truth table, the minterm $X \cdot Y \cdot Z'$ is 1 only in row 6 of the truth table and the minterm $X \cdot Y \cdot Z$ is 1 only in row 7 of the truth table; (I hope you understand it).

Regarding the maxterms now, the maxterm $X+Y+Z$ is 0 only in row 0 of the truth table. In row 0 of the truth table $X=0$, $Y=0$, $Z=0$, so $X+Y+Z = 0+0+0 = 0$. The maxterm $X+Y+Z'$ is 0 only in row 1 of the truth table. In row 1 of the truth table $X=0$, $Y=0$, $Z=1$, so $Z'=0$ and $X+Y+Z' = 0+0+0 = 0$. The maxterm $X+Y'+Z$ is 0 only in row 2 of the truth table. In row 2 of the truth table $X=0$, $Y=1$, $Z=0$, so $Y'=0$ and $X+Y'+Z = 0+0+0 = 0$. Similarly, the maxterm $X+Y'+Z'$ is 0 only in row 3 of the truth table, the maxterm $X'+Y+Z$ is 0 only in row 4 of the truth table, the maxterm $X'+Y+Z'$ is 0 only in row 5 of the truth table, the maxterm $X'+Y'+Z$ is 0 only in row 6 of the truth table and the maxterm $X'+Y'+Z'$ is 0 only in row 7 of the truth table.

Observation: An interesting observation is that for each row in the truth table, the minterm and the corresponding maxterm are complements of each other. Apply De Morgan's theorems (T13), (T13') to see that. Refer to table 2 on page 5. Consider row 0. The minterm is $X' \cdot Y' \cdot Z'$. Take the complement of this minterm using theorem (T13). What you get is $(X' \cdot Y' \cdot Z')' = (X')' + (Y')' + (Z')' = X + Y + Z$ which is the maxterm corresponding to row 0. As another example consider row 4. The minterm corresponding to row 4 is $X \cdot Y' \cdot Z'$. Take the complement of this minterm. What you get is $(X \cdot Y' \cdot Z')' = X' + (Y')' + (Z')' =$ $= X' + Y + Z$ which is the maxterm corresponding to row 4. You can do the rest by yourselfs.

- Minterm number: A minterm number is an n-bit integer used to represent an n-variable minterm. The name minterm i will be used to denote the minterm corresponding to row i of the truth table. For example, minterm 6 is $X \cdot Y \cdot Z'$, minterm 4 is $X \cdot Y' \cdot Z'$ (see table 2 on page 5). In minterm i, a particular variable appears complemented if the corresponding bit in the binary representation of i is 0; otherwise it is uncomplemented. Consider for example the minterm 4 of table 2 of page 5. The binary representation of 4 (3-bit representation because we have three variables) is 100. Thus, the minterm 4 is $X \cdot Y' \cdot Z'$

• <u>Maxterm number:</u> A maxterm number is an n-bit integer used to represent an n-variable maxterm. The name <u>maxterm i</u> will be used to denote the maxterm corresponding to row i of the truth table. For example, maxterm 3 is $X+Y'+Z'$, maxterm 7 is $X'+Y'+Z'$ (see table 2 on page 5). In maxterm i, a particular variable appears complemented if the corresponding bit in the binary representation of i is 1; otherwise it is uncomplemented. Consider for example the maxterm 6 of table 2 of page 5. The 3-bit binary representation of 6 is 110. Thus, the maxterm 6 is $X'+Y'+Z$.

— <u>Based on the correspondence between the truth table and minterms, one can easily create an algebraic representation of a logic function from its truth table.</u>

Let me demonstrate this with the example of the function F of table 1 on page 3 of this handout. As seen from table 1 function F is 1 if ($X=0$ and $Y=0$ and $Z=0$) or ($X=0$ and $Y=1$ and $Z=1$) or ($X=1$ and $Y=0$ and $Z=0$) or ($X=1$ and $Y=1$ and $Z=0$) or ($X=1$ and $Y=1$ and $Z=1$). In other words, function F is 1 if (minterm 0 is 1) or (if ~~is~~ minterm 3 is 1) or (if minterm 4 is 1) or (if minterm 6 is 1) or (if minterm 7 is 1). In other words, function F is 1 if $X'\cdot Y'\cdot Z'=1$ or $X'\cdot Y\cdot Z=1$ or $X\cdot Y'\cdot Z'=1$ or $X\cdot Y\cdot Z'=1$ or $X\cdot Y\cdot Z=1$. This can be written as a logic equation shown on the next page.

$$F = \Sigma_{X,Y,Z}(0,3,4,6,7) =$$
$$= X' \cdot Y' \cdot Z' + X' \cdot Y \cdot Z + X \cdot Y' \cdot Z' + X \cdot Y \cdot Z' + X \cdot Y \cdot Z \quad (1)$$

– In the above equation (1), the notation $\Sigma_{X,Y,Z}(0,3,4,6,7)$ is a minterm list and means the sum of minterms 0, 3, 4, 6, 7 with variables X, Y, Z.

- Canonical sum: The canonical sum of a logic function is a sum of the minterms corresponding to truth table rows (input combinations) for which the function produces a 1 output. For example, equation (1) above shows the canonical sum for the function F of table 1 of page 3 of this handout.

– Based on the correspondence between the truth table and maxterms, one can easily create an algebraic representation of a logic function from its truth table based on maxterms ~~this time~~:

Let me demonstrate this with the example of function F of table 1 on page 3 of this handout. As seen from table 1, function F is 0 if (X=0 and Y=0 and Z=1) or (X=0 and Y=1 and Z=0) or (X=1 and Y=0 and Z=1). Applying DeMorgan's theorem on the above statemement we get: Function F is 1 if (X=1 or Y=1 or Z=0) and (X=1 or Y=0 or Z=1) and (X=0 or Y=1 or Z=0). As you see when I applied DeMorgan's theorem I swapped 0's and 1's and or and and. In other words, function F is 1 if (maxterm 1 is 1) and (maxterm 2 is 1) and (maxterm 5 is 1)

In other words, function F is 1 if $X+Y+Z'=1$ and $X+Y'+Z=1$ and $X'+Y+Z'=1$. This can be written as a logic equation as shown below:

$$F = \prod_{X,Y,Z}(1,2,5) = \qquad (2).$$
$$= (X+Y+Z')\cdot(X+Y'+Z)\cdot(X'+Y+Z')$$

– In the above equation (2), the notation $\prod_{X,Y,Z}(1,2,5)$ is a <u>maxterm list</u> and means the product of maxterms 1, 2, 5 with variables X, Y, Z.

• <u>Canonical product:</u> The canonical product of a logic function is a product of the maxterms corresponding to input combinations for which the function produces a 0 output. For example, equation (2) above shows the canonical product for the function F of table1 of page 3 of this handout.

• <u>Converting between a minterm list and a maxterm list:</u>

It is <u>very</u> easy to convert between a minterm list and a maxterm list. For a function of <u>n</u> variables, the possible minterm and maxterm numbers will be in the set $\{0,1,2,3,\cdots,2^n-1\}$. A minterm or maxterm list contains a subset of these numbers.

~~Therefore, to switch between list types take the set complement. Some examples are shown in next page.~~

To switch between list types, take the set complement. Some examples are shown below:

- $\sum_{X,Y,Z}(1,4,7) = \prod_{X,Y,Z}(0,2,3,5,6)$
- $\prod_{X,Y}(1,2) = \sum_{X,Y}(0,3)$
- $\sum_{A,B,C}(0,1,2,3) = \prod_{A,B,C}(4,5,6,7)$
- $\sum_{X,Y}(1) = \prod_{X,Y}(0,2,3)$
- $\sum_{W,X,Y,Z}(0,1,2,3,5,7,11,13) = \prod_{W,X,Y,Z}(4,6,8,9,10,12,14,15)$

Example: Consider the function $F(A,B,C)$ provided by the truth table below; (it is table 3). Write $F(A,B,C)$ in canonical sum form and canonical product form.

A B C	F
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Table 3

Answer: The canonical sum form is

$$F = \sum_{A,B,C}(3,4,5,6,7) =$$

$$= A'\cdot B\cdot C + A\cdot B'\cdot C' + A\cdot B'\cdot C + A\cdot B\cdot C' + A\cdot B\cdot C$$

The canonical product form is

$$F = \prod_{A,B,C}(0,1,2) = (A+B+C)\cdot(A+B+C')\cdot(A+B'+C).$$

Example: Write in canonical sum form and canonical product form the function $F(a,b,c,d) = a'\cdot(b'+d)+a\cdot c\cdot d'$

Answer: I will first provide the canonical sum form of the function. We have $F = a'\cdot(b'+d)+a\cdot c\cdot d' =$

$$= a'\cdot b' + a'\cdot d + a\cdot c\cdot d' = a'\cdot b'\cdot\underbrace{(c+c')}_{1}\cdot\underbrace{(d+d')}_{1} +$$

$$+ a'\cdot d\cdot\underbrace{(b+b')}_{1}\cdot\underbrace{(c+c')}_{1} + a\cdot c\cdot d'\cdot\underbrace{(b+b')}_{1} =$$

$$= (a'\cdot b'\cdot c + a'\cdot b'\cdot c')\cdot(d+d') + (a'\cdot d\cdot b + a'\cdot d\cdot b')\cdot(c+c') +$$

$$+ a\cdot c\cdot d'\cdot b + a\cdot c\cdot d'\cdot b' = \underline{a'\cdot b'\cdot c\cdot d} + a'\cdot b'\cdot c\cdot d' + \underline{a'\cdot b'\cdot c'\cdot d}$$

$$+ a'\cdot b'\cdot c'\cdot d' + a'\cdot d\cdot b\cdot c + a'\cdot d\cdot b\cdot c' + \underbrace{a'\cdot d\cdot b'\cdot c} + \underline{a'\cdot d\cdot b'\cdot c'} +$$

$$+ a\cdot c\cdot d'\cdot b + a\cdot c\cdot d'\cdot b' = a'\cdot b'\cdot c\cdot d + a'\cdot b'\cdot c\cdot d + a'\cdot b'\cdot c\cdot d' +$$

$$a'\cdot b'\cdot c'\cdot d + a'\cdot b'\cdot c'\cdot d + a'\cdot b'\cdot c'\cdot d' + a'\cdot b\cdot c\cdot d + a'\cdot b\cdot c'\cdot d +$$

$$a\cdot b\cdot c\cdot d' + a\cdot b'\cdot c\cdot d' = a'\cdot b'\cdot c\cdot d + a'\cdot b'\cdot c\cdot d' + a'\cdot b'\cdot c'\cdot d$$

$$+ a'\cdot b'\cdot c'\cdot d' + a'\cdot b\cdot c\cdot d + a'\cdot b\cdot c'\cdot d + a\cdot b\cdot c\cdot d' + a\cdot b'\cdot c\cdot d'$$

$$= \underset{0\ 0\ 0\ 0}{a'\cdot b'\cdot c'\cdot d'} + \underset{0\ 0\ 0\ 1}{a'\cdot b'\cdot c'\cdot d} + \underset{0\ 0\ 1\ 0}{a'\cdot b'\cdot c\cdot d'} + \underset{0\ 0\ 1\ 1}{a'\cdot b'\cdot c\cdot d} +$$

$$+ \underset{0\ 1\ 0\ 1}{a'\cdot b\cdot c'\cdot d} + \underset{0\ 1\ 1\ 1}{a'\cdot b\cdot c\cdot d} + \underset{1\ 0\ 1\ 0}{a\cdot b'\cdot c\cdot d'} + \underset{1\ 1\ 1\ 0}{a\cdot b\cdot c\cdot d'} =$$

$$\Sigma_{a,b,c,d}(0,1,2,3,5,7,10,14).$$

The above is the canonical sum for the given function F.

We now provide the canonical product for the function F. It is

$$F = \prod_{a,b,c,d}(4,6,8,9,11,12,13,15) =$$

$$= (a+b'+c+d)\cdot(a+b'+c'+d)\cdot(a'+b+c+d)\cdot(a'+b+c+d')$$
$$(a'+b+c'+d')\cdot(a'+b'+c+d)\cdot(a'+b'+c+d')\cdot(a'+b'+c'+d')$$

<u>Example:</u> Prove that $a'\cdot c+b'\cdot c'+a\cdot b = a'\cdot b'+b\cdot c+a\cdot c'$

<u>Answer:</u> I will find the canonical sums of both the left and the right side of the above equation and show that they are equal.

For the left side we have: $a'\cdot c+b'\cdot c'+a\cdot b =$

$$= a'\cdot c\cdot\underbrace{(b+b')}_{1} + b'\cdot c'\cdot\underbrace{(a+a')}_{1} + a\cdot b\cdot\underbrace{(c+c')}_{1} =$$

$$= a'\cdot c\cdot b + a'\cdot c\cdot b' + b'\cdot c'\cdot a + b'\cdot c'\cdot a' + a\cdot b\cdot c + a\cdot b\cdot c'$$

$$= a'\cdot b'\cdot c' + a'\cdot b'\cdot c + a'\cdot b\cdot c + a\cdot b'\cdot c' + a\cdot b\cdot c' + a\cdot b\cdot c$$

$$= \sum_{a,b,c}(0,1,3,4,6,7).$$

For the right side we have: $a'\cdot b'+b\cdot c+a\cdot c' =$

$$a'\cdot b'\underbrace{(c+c')}_{1} + b\cdot c\cdot\underbrace{(a+a')}_{1} + a\cdot c'\cdot\underbrace{(b+b')}_{1} =$$

$$a'\cdot b'\cdot c + a'\cdot b'\cdot c' + b\cdot c\cdot a + b\cdot c\cdot a' + a\cdot c'\cdot b + a\cdot c'\cdot b'$$

$$= a'\cdot b'\cdot c' + a'\cdot b'\cdot c + a'\cdot b\cdot c + a\cdot b'\cdot c' + a\cdot b\cdot c' + a\cdot b\cdot c$$

$$= \sum_{a,b,c}(0,1,3,4,6,7)$$

Because the canonical sums of both left and right side of the equation are equal, the equation is valid.

- Review

What you have learned so far is five possible representations for a combinational logic function:

1. A truth table.
2. An algebraic sum of minterms, the canonical sum.
3. A minterm list using the Σ notation.
4. An algebraic product of maxterms, the canonical product.
5. A maxterm list using the Π notation.

Each one of these representations specifies exactly the same information. Given any one of them, you can derive the other four.