EE 2720

Handout #6

<u>Definition</u>: A <u>literal</u> is a variable or the complement of a variable. For example, X, Y, Z, X', Y', Z'.

- <u>Algebraic Simplification of Switching Expressions</u>

Here we present methods for simplifying switching expressions, using the theorems of switching algebra. There are three basic ways of simplifying switching functions. These are: <u>Combining terms</u>, <u>eliminating terms</u>, and <u>eliminating literals</u>.

1. <u>Combining terms</u>: Use the theorem (T10) or $X\cdot Y + X\cdot Y' = X$ to combine two terms. For example $a\cdot b\cdot c'\cdot d' + a\cdot b\cdot c\cdot d' = a\cdot b\cdot d'$. Here we assigned $X = a\cdot b\cdot d'$ and $Y = c$. Another example follows: Simplify $a\cdot b'\cdot c + a\cdot b\cdot c + a'\cdot b\cdot c$. This expression becomes $a\cdot b'\cdot c + a\cdot b\cdot c + a\cdot b\cdot c + a'\cdot b\cdot c = a\cdot c + b\cdot c$. In the above example I also used theorem (T3) which states $X + X = X$.
The theorem (T10) can still be used when X and Y are replaced with more complicated expressions. For example
$(a+b\cdot c)\cdot(d+e') + a'\cdot(b'+c')\cdot(d+e') = d+e'$. Here we assigned $X = d+e'$, $Y = a+b\cdot c$, $Y' = a'\cdot(b'+c')$. So, the above expression becomes $Y\cdot X + Y'\cdot X = X\cdot Y + X\cdot Y' =$
$= X = d+e'$.

2. <u>Eliminating terms</u>: Use the theorem (T9) or $X + X\cdot Y = X$ to eliminate redundant terms if possible. Then try to apply the consensus theorem (T11) or $X\cdot Y + X'\cdot Z + Y\cdot Z = X\cdot Y + X'\cdot Z$ to eliminate

any consensus terms. See examples below.

$a' \cdot b + a' \cdot b \cdot c = a' \cdot b$. Here we assigned $X = a' \cdot b$ and $Y = c$ so the expression becomes $X + X \cdot Y = X =$ $= a' \cdot b$.

Another example follows

$$a' \cdot b \cdot c' + b \cdot c \cdot d + a' \cdot b \cdot d = c' \cdot a' \cdot b + c \cdot b \cdot d + a' \cdot b \cdot b \cdot d =$$

$$= \underbrace{c}_{X} \cdot \underbrace{b \cdot d}_{Y} + \underbrace{c'}_{X'} \cdot \underbrace{a' \cdot b}_{Z} + \underbrace{b \cdot d}_{Y} \cdot \underbrace{a' \cdot b}_{Z} = \underbrace{c}_{X} \cdot \underbrace{b \cdot d}_{Y} + \underbrace{c'}_{X'} \cdot \underbrace{a' \cdot b}_{Z}$$

according to the consensus theorem (T11). Here I also used some other theorems as well. Which are they?

3. Eliminating literals: Use the theorem $X + X' \cdot Y =$ $= X + Y$ to eliminate redundant literals. The above theorem is the same as theorem (T11'') which states $X \cdot Y' + Y = X + Y$. Here we just substituted X with Y and Y with X. Simple factoring (theorem T8) may be needed before the theorem is applied. See example.

$$A' \cdot B + A' \cdot B' \cdot C' \cdot D' + A \cdot B \cdot C \cdot D' = A' \cdot (B + B' \cdot C' \cdot D') + A \cdot B \cdot C \cdot D' =$$

$$= A' \cdot (B + C' \cdot D') + A \cdot B \cdot C \cdot D' = A' \cdot B + A' \cdot C' \cdot D' + A \cdot B \cdot C \cdot D'$$

$$= A' \cdot B + A \cdot B \cdot C \cdot D' + A' \cdot C' \cdot D' = B \cdot (A' + A \cdot C \cdot D') + A' \cdot C' \cdot D'$$

$$= B \cdot (A' + C \cdot D') + A' \cdot C' \cdot D' = A' \cdot B + B \cdot C \cdot D' + A' \cdot C' \cdot D'.$$

Here I also used theorem (T8) $(X \cdot Y + X \cdot Z = X \cdot (Y + Z))$ and some other theorems.

Another example follows. This example is based on the method of adding redundant terms. Redundant terms can be introduced in several ways such as adding the consensus term, adding $X \cdot X'$, multiplying by $(X + X')$ or adding $X \cdot Y$ to X (see theorem T9). The added terms should be chosen so that they will combine with or eliminate other terms. The example follows:

$\underline{W\cdot X}+X\cdot Y+\underline{X'\cdot Z'}+W\cdot Y'\cdot Z'$ (add $W\cdot Z'$ by consensus theorem)

$= W\cdot X+X\cdot Y+X'\cdot Z'+W\cdot Y'\cdot Z'+W\cdot Z' =$

$= W\cdot X+X\cdot Y+X'\cdot Z'+\underbrace{W\cdot Z'}_{A}+\underbrace{W\cdot Z'}_{A}\cdot\underbrace{Y'}_{B} =$

$= \underline{X\cdot W}+X\cdot Y+\underline{X'\cdot Z'}+\underline{W\cdot Z'}$. (eliminate $W\cdot Z'$ by consensus theorem)

$= X\cdot W+X\cdot Y+X'\cdot Z'$.

• Multiplying out and Factoring

Some definitions follow first.

- Product term: A product term is a single literal or a logical product of two or more literals: Examples: Z', $W\cdot X\cdot Y$, $X\cdot Y'\cdot Z$, $W'\cdot Y'\cdot Z$.
- Sum-of-products expression: A sum-of-products expression is a logical sum of product terms. Examples: $A\cdot B'+C\cdot D'\cdot E+A\cdot C'\cdot E$, $A\cdot B\cdot C'+D\cdot E\cdot F\cdot G+H$, $A+B'+C+ +D'\cdot E$.
- Sum term: A sum term is a single literal or a logical sum of two or more literals. Examples: Z', $W+X+Y$, $X+Y'+Z$, $W'+Y'+Z$.
- Product-of-sum expression: A product-of-sum expression is a logical product of sum terms. Examples: $(A+B')\cdot(C+D'+E)\cdot(A+C'+E')$, $(A+B)\cdot(C+D+E)\cdot F$.

We now present the subject of multiplying out and factoring.

• Multiplying out

By multiplying out we mean transforming an expression from product-of-sum form into the corresponding sum-of-products form. The theorems used for multiplying out are the following:

(T8) $X\cdot(Y+Z)=X\cdot Y+X\cdot Z$

(T8') $(X+Y)\cdot(X+Z)=X+Y\cdot Z$

In addition, the following theorem is very useful for multiplying out:

$$(X+Y)\cdot(X'+Z)=X\cdot Z+X'\cdot Y \qquad (1)$$

Since the above theorem of equation (1) is new, we now prove it.

Proof:

If $X=0$, equation (1) reduces to $Y\cdot(1+Z)=0+1\cdot Y$ or

$Y\cdot 1=1\cdot Y$ or $Y=Y$.

If $X=1$, equation (1) becomes $(1+Y)\cdot Z=1\cdot Z+0\cdot Y$ or $1\cdot Z=1\cdot Z$ or $Z=Z$.

Because equation (1) is valid for both $X=0$ and $X=1$, it is always valid and the proof is completed.

Note: To avoid generating unnecessary terms when multiplying out, theorem (T8') and theorem of equation (1) should generally be applied before theorem (T8) is applied.

Some examples of multiplying out follow on the next page.

<u>Example</u>: Multiply out $(A+B\cdot C)\cdot(A+D+E)$.

<u>Answer:</u> Let $X=A$, $Y=B\cdot C$, $Z=D+E$. I first apply theorem (T8'). The above expression becomes $(X+Y)\cdot(X+Z)=X+Y\cdot Z=A+B\cdot C\cdot(D+E)=$
$=A+B\cdot C\cdot D+B\cdot C\cdot E$ (after applying theorem (T8)).

<u>Example:</u> Multiply out $(Q+A\cdot B')\cdot(Q'+C'\cdot D)$.

<u>Answer:</u> Let $X=Q$, $Y=A\cdot B'$, $Z=C'\cdot D$. Then the given expression becomes $(X+Y)\cdot(X'+Z)=$
$=X\cdot Z+X'\cdot Y=Q\cdot C'\cdot D+Q'\cdot A\cdot B'$; (Here I applied the theorem of equation (1)).

If we simply multiplied out by ~~using~~ theorem (T8), we would get three terms (actually four) instead of two as shown below:

$(Q+A\cdot B')\cdot(Q'+C'\cdot D)=Q\cdot Q'+Q\cdot C'\cdot D+A\cdot B'\cdot Q'+A\cdot B'\cdot C'\cdot D$
$=0+Q\cdot C'\cdot D+Q'\cdot A\cdot B'+A\cdot B'\cdot C'\cdot D=$
$=Q\cdot C'\cdot D+Q'\cdot A\cdot B'+A\cdot B'\cdot C'\cdot D$. Because the term $A\cdot B'\cdot C'\cdot D$ is difficult to eliminate, it is much better to use the theorem of equation (1) instead of theorem (T8).

<u>Example:</u> Multiply out
$(A+B+C')\cdot(A+B+D)\cdot(A+B+E)\cdot(A+D'+E)\cdot(A'+C)$

<u>Answer:</u> Let $X=A+B$, $Y=C'$, $Z=D$, $F=A$, $G=D'+E$.

Then after applying theorem (T8') and theorem of equation (1) the above expression becomes

$$(X+Y)\cdot(X+Z)\cdot(X+E)\cdot(F+G)\cdot(F'+Y') =$$
$$= (X+\underbrace{Y\cdot Z}_{K})\cdot(X+E)\cdot(F\cdot Y' + F'\cdot G) =$$
$$= (X+K)\cdot(X+E)(F\cdot Y' + F'\cdot G) =$$
$$= (X+K\cdot E)\cdot(F\cdot Y' + F'\cdot G) = (X+Y\cdot Z\cdot E)\cdot(F\cdot Y' + F'\cdot G) =$$
$$= (A+B+C'\cdot D\cdot E)\cdot[A\cdot C + A'\cdot(D'+E)] =$$
$$= (A+B+C'\cdot D\cdot E)\cdot(A\cdot C + A'\cdot D' + A'\cdot E) =$$
$$= A\cdot A\cdot C + A\cdot A'\cdot D + A\cdot A'\cdot E + B\cdot A\cdot C + B\cdot A'\cdot D' +$$
$$+ B\cdot A'\cdot E + C'\cdot D\cdot E\cdot A\cdot C + C'\cdot D\cdot E\cdot A'\cdot D' + C'\cdot D\cdot E\cdot A'\cdot E$$
$$= A\cdot C + 0 + 0 + A\cdot B\cdot C + A'\cdot B\cdot D' + A'\cdot B\cdot E + 0 + 0 +$$
$$+ A'\cdot C'\cdot D\cdot E = A\cdot C + A\cdot B\cdot C + A'\cdot B\cdot D' + A'\cdot B\cdot E +$$
$$+ A'\cdot C'\cdot D\cdot E = A\cdot C\cdot 1 + A\cdot B\cdot C + A'\cdot B\cdot D' + A'\cdot B\cdot E$$
$$+ A'\cdot C'\cdot D\cdot E = A\cdot C\cdot(1+B) + A'\cdot B\cdot D' + A'\cdot B\cdot E + A'\cdot C'\cdot D\cdot E$$
$$= A\cdot C\cdot 1 + A'\cdot B\cdot D' + A'\cdot B\cdot E + A'\cdot C'\cdot D\cdot E =$$
$$= A\cdot C + A'\cdot B\cdot D' + A'\cdot B\cdot E + A'\cdot C'\cdot D\cdot E.$$

Here we also applied theorem (T8) and some other theorems. Which are they?

- In this example that we just now presented, if only theorem (T8) had been used to multiply out, 162 terms would have been generated, and 158 of these terms would then have to be eliminated. This would be a lot of trouble; (it would have taken a lot of time).

- <u>Factoring</u>

By factoring we mean transforming an expression from sum-of-product form into the corresponding product-of-sum form. Again, the theorems used for factoring are the same ones as these for multiplying out or theorem (T8), theorem (T8′) and theorem of equation (1). Some examples of factoring follow:

<u>Example:</u> Factor $A+B'\cdot C\cdot D$.

<u>Answer:</u> Let $X=A$, $Y=B'$, $Z=C\cdot D$. Then the above expression becomes according to theorem (T8′)

$$X+Y\cdot Z=(X+Y)\cdot(X+Z)=(A+B')\cdot(A+C\cdot D).$$

Now apply theorem (T8′) again. The expression finally becomes

$$(A+B')\cdot(A+C)\cdot(A+D).$$

<u>Example:</u> Factor $A\cdot B'+C'\cdot D$.

<u>Answer:</u> Apply theorem (T8′) to get

$$\underbrace{A\cdot B'}_{X}+C'\cdot D = X+C'\cdot D=(X+C')\cdot(X+D)=$$

$$=(A\cdot B'+C')\cdot(A\cdot B'+D)=(C'+A\cdot B')\cdot(D+A\cdot B').$$

Apply theorem (T8′) again to get

$$(C'+A)\cdot(C'+B')\cdot(D+A)\cdot(D+B').$$

<u>Example:</u> Factor $C'\cdot D+C'\cdot E'+G'\cdot H$.

<u>Answer:</u> Here we apply theorem (T8) first. The expression then becomes: $C'\cdot D+C'\cdot E'+G'\cdot H=$

$$=C'\cdot(D+E')+G'\cdot H=G'\cdot H+C'\cdot(D+E').$$

Now let $X=G'\cdot H$, $Y=C'$, $Z=D+E'$ and apply theorem (T8′). The expression then becomes

$X + Y \cdot Z = (X+Y) \cdot (X+Z) = (G' \cdot H + C') \cdot (G' \cdot H + D + E')$
$= (C' + G' \cdot H) \cdot (D + E' + G' \cdot H)$. Apply again theorem (T8') to finally get
$(C' + G' \cdot H) \cdot (D + E' + G' \cdot H) =$
$= (C' + G') \cdot (C' + H) \cdot (D + E' + G') \cdot (D + E' + H)$.

<u>Example:</u> Factor $A \cdot C + A' \cdot B \cdot D' + A' \cdot B \cdot E + A' \cdot C' \cdot D \cdot E$

<u>Answer:</u> Here we apply theorem (T8) first. The expression then becomes
$A \cdot C + A' \cdot B \cdot D' + A' \cdot B \cdot E + A' \cdot C' \cdot D \cdot E =$
$A \cdot C + A' \cdot (B \cdot D' + B \cdot E + C' \cdot D \cdot E)$.
Now let $X = A$, $Y = B \cdot D' + B \cdot E + C' \cdot D \cdot E$, $Z = C$ and apply the theorem of equation (1). The above expression then becomes $X \cdot Z + X' \cdot Y =$
$= (X+Y) \cdot (X'+Z) =$
$= (A + B \cdot D' + B \cdot E + C' \cdot D \cdot E) \cdot (A' + C)$. Apply theorem (T8) again to get
$[A + C' \cdot D \cdot E + B \cdot (D' + E)] \cdot (A' + C)$.

Now let $F = A + C' \cdot D \cdot E$, $G = D' + E$ and apply theorem (T8'). The above expression then becomes $(F + B \cdot G)(A' + C) = (F + B) \cdot (F + G) \cdot (A' + C)$
$= (A + C' \cdot D \cdot E + B) \cdot (A + C' \cdot D \cdot E + D' + E) \cdot (A' + C)$
$= (A + B + C' \cdot D \cdot E) \cdot (A + E \cdot 1 + C' \cdot D \cdot E + D') \cdot (A' + C) =$
$= (A + B + C' \cdot D \cdot E) \cdot [A + E \cdot (1 + C' \cdot D) + D'] \cdot (A' + C) =$
$= (A + B + C' \cdot D \cdot E) \cdot (A + E \cdot 1 + D') \cdot (A' + C) =$

$$= (A+B+C'\cdot D\cdot E)\cdot(A+E+D')\cdot(A'+C).$$

Now let $H = A+B$ and $J = D\cdot E$ and apply theorem (T8'). The above expression becomes

$$(H+C'\cdot J)\cdot(A+E+D')\cdot(A'+C) =$$

$$= (H+C')\cdot(H+J)\cdot(A+E+D')\cdot(A'+C) =$$

$$= (A+B+C')\cdot(A+B+D\cdot E)\cdot(A+E+D')\cdot(A'+C)$$

Applying againg theorem (T8') to the second term from the left ($A+B+D\cdot E$ I mean) we finelly get

$$(A+B+C')\cdot(A+B+D)\cdot(A+B+E)\cdot(A+E+D')\cdot(A'+C).$$