EE 2720

Handout #5

• Boolean Algebra and Switching Algebra

• In 1854, George Boole created a two valued algebraic system called Boolean algebra.

• In 1938, Claude Shannon adapted Boolean algebra to analyze and describe the behavior of circuits built with relays. This adaptation is called switching algebra.

• Switching Algebra

In switching algebra the condition of a logic signal is represented by symbolic variables, such as X, Y and these variables can only have two values, 0 or 1.
There are two possible conventions; positive logic and negative logic.
— For the positive logic, we assign 0 to the LOW signal and 1 to the HIGH signal.

— For the negative logic, we assign 1 to the LOW signal and 0 to the HIGH signal.

• Axioms

The axioms of a mathematical system are a minimum set of basic definitions that are assumed to be true, and from which all other information about the system can be derived.
The first two axioms of switching algebra state that a variable X can take only one of two values:

$$\boxed{(A1)\ X=0 \text{ if } X\neq 1\ ;\ (A1')\ X=1 \text{ if } X\neq 0.}$$

As we said earlier in the semester (very beginning of the semester) one of the three most basic digital devices is the NOT gate or inverter; (see Figure 1-1 of page 7 of text).
A NOT gate or inverter is a digital device whose output signal is the opposite (or complement) of its input signal. This means that if the input is 0 then the output is 1 and if the input is 1 then the output is 0.
A prime (′) will be used to denote the inverter's function. This is to say that if variable X denotes an inverter's input signal, then

X' denotes the value of a signal on the inverter's output.

We now have the second pair of axioms:

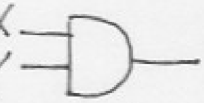
$\boxed{\text{(A2) If } X=0, \text{ then } X'=1 \qquad \text{(A2') If } X=1, \text{ then } X'=0.}$

A NOT gate or inverter is shown below

X  $Y = X' = \text{NOT } X$

<u>Note</u>: X' can read as "X prime" or "NOT X."

As said earlier in the semester (very beginning of the semester) another one of the three most basic digital devices is the <u>2-input AND gate</u>; (see Figure 1-1 on page 7 of text). The 2-input AND gate is a device whose output is 1 if both its inputs are 1. The function of a 2-input AND gate is also called <u>logical multiplication</u> denoted by a multiplication dot $(\cdot)$. This is to say that an AND gate with inputs X and Y has an output signal whose value is $X \cdot Y$. This is shown by the figure below

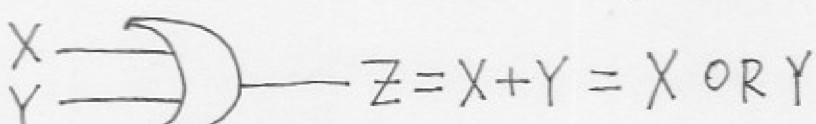
X, Y  $Z = X \cdot Y = X \text{ AND } Y$

AND gate

We now have the next three axioms that state the definition of the AND operation by listing the output produced by the AND gate for each possible input combination.

(A3) $0 \cdot 0 = 0$
(A4) $1 \cdot 1 = 1$
(A5) $0 \cdot 1 = 1 \cdot 0 = 0$

Again as we said at the very beginning of the semester, the last of the three most basic digital devices is the <u>2-input OR gate</u>; (see Figure 1-1 on page 7 of text). The 2-input OR gate is a device whose output is 1 if either of its inputs or both are 1. The function of a 2-input OR gate is also called <u>logical addition</u> denoted by a plus sign (+). This is to say that an OR gate with inputs X and Y has an output signal whose value is X+Y. This is shown by the figure below.



OR gate

We now have the last three axioms that state the definition of the OR operation by listing the output produced by the OR gate for each possible input combination.

$$\boxed{\begin{array}{l} (A3') \ 1+1=1 \\ (A4') \ 0+0=0 \\ (A5') \ 1+0=0+1=1 \end{array}}$$

<u>IMPORTANT</u>: In the above, + indicates <u>logical addition</u> and <u>not</u> <u>arithetic addition</u>. So

$1+1=1$ and it is <u>NOT</u> $1+1=2$; (this is a common mistake that many students do).

<u>Note</u>: The above stated axioms (A1)-(A5) and (A1')-(A5') completely define switching algebra. All other facts can be proved using these axioms.

• Precedence

The precedence of operations in a logic expression is the following:; (by precedence we mean the order in which the operations are performed).

1. Parentheses.
2. Complement.
3. Logical multiplication; (AND operation).
4. Logical addition; (OR operation).

Note: In the previous page, lower numbered operations have higher precedence and are performed first. For example, expressions inside parentheses are performed first, followed by complement operations, followed by AND operations, followed by OR operations.

• Switching-algebra theorems

Switching-algebra theorems are statements known to be always true, that are used to manipulate algebraic expressions to allow simpler analysis or more efficient synthesis of circuits.

I will now present swithing-algebra theorems with one variable, switching-algebra theorems with two or three variables and switching-algebra theorems with n variables.

Note: The theorems with n variables can be proved with the so called finite induction technique. With this technique, you first prove that the theorem is true for the case where $n=2$ (basis step), then you prove that if the theorem is true for $n=i$, then it is also true for $n=i+1$ (induction step).

Note: Another way of proving a theorem with any number of variables is by using a so called truth table. Here you should have a finite number of variables.

<u>Definition</u>: A truth table (also called a table of combinations) specifies the values of a Boolean expression for every possible combination of values of the variables in the expression. If an expression has n-variables, and each variable can take the value 0 or 1, the number of different combinations of values of the variables is

$$\underbrace{2\times2\times2\times\cdots\times2\times2}_{n \text{ times}} = 2^n$$

Therefore, a truth table for an n-variable expression will have 2^n rows; (examples will be shown later).

The switching-algebra theorems now follow:

- <u>Switching-algebra theorems with one variable</u>

Here the single variable is X. The theorems follow.

(T1) $X+0=X$	(T1′) $X\cdot1=X$	(Identities)
(T2) $X+1=1$	(T2′) $X\cdot0=0$	(Null elements)
(T3) $X+X=X$	(T3′) $X\cdot X=X$	(Idempotency)
(T4) $(X')'=X$		(Involution)
(T5) $X+X'=1$	(T5′) $X\cdot X'=0$	(Complements)

- Proof of theorem (T2) or proof of $X+1=1$.

  Proof: We have two cases:

  Case $X=0$
  - $0+1=1$ according to axiom (A5′)

  Case $X=1$
  - $1+1=1$ according to axiom (A3′)

- Proof of theorem (T1): Look on page 198 of text for a proof.

- Proof of theorem (T5) or proof of $X+X'=1$.

  Proof: We have two cases
  - Case $X=0$. Then $X'=1$ according to axiom (A2). Therefore $X+X'=0+1=1$ according to axiom (A5′).
  - Case $X=1$. Then $X'=0$ according to axiom (A2′). Therefore $X+X'=1+0=1$ according to axiom (A5′).

Do the rest of the proofs by yourselfs.

Example: Simplify the expression $(A\cdot B'+D)\cdot E+1$.

Answer: Let X be $X=(A\cdot B'+D)\cdot E$. Then the given expression becomes $X+1$ and according to theorem (T2) $X+1=1$. Therefore $(A\cdot B'+D)\cdot E+1=1$.

Example: Simplify the expression
$(A\cdot B'+D)\cdot(A\cdot B'+D)'$.

Answer: Let X be $X=A\cdot B'+D$. Then the given expression becomes $X\cdot X'$ and according to theorem (T5') $X\cdot X'=0$. Therefore, $(A\cdot B'+D)\cdot(A\cdot B'+D)'=0$.

- Switching-algebra theorems with two or three variables

These theorems follow.

(T6) $X+Y=Y+X$ (T6') $X\cdot Y=Y\cdot X$ (Commutativity)
(T7) $(X+Y)+Z=X+(Y+Z)$ (T7') $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$ (Associativity)
(T8) $X\cdot Y+X\cdot Z=X\cdot(Y+Z)$ (T8') $(X+Y)\cdot(X+Z)=X+Y\cdot Z$ (Distributivity).
(T9) $X+X\cdot Y=X$ (T9') $X\cdot(X+Y)=X$ (Covering).
(T10) $X\cdot Y+X\cdot Y'=X$ (T10') $(X+Y)\cdot(X+Y')=X$ (Combining).
(T11) $X\cdot Y+X'\cdot Z+Y\cdot Z=X\cdot Y+X'\cdot Z$
(T11') $(X+Y)\cdot(X'+Z)\cdot(Y+Z)=(X+Y)\cdot(X'+Z)$ (Consensus)

- Comments: Theorems (T9) and (T10) are used in minimization of logic functions. Also (T11) is very useful in simplifying logic expressions

- Proof of theorem (T9): see page 200 of text for a proof.

- Proof of theorem (T10):

  Proof

  $$X \cdot Y + X Y' = X \cdot (Y + Y') \quad \text{according to (T8)}$$
  $$= X \cdot 1 \quad \text{according to (T5)}$$
  $$= X \quad \text{according to (T1')}.$$

- Proof of theorem (T7') or proof of
  $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ (1)

Proof: We will prove equation (1) using a truth table. On the left side of the table, we will list all possible combinations of values of the variables X, Y and Z. The number of different combinations of values of the variables is $2^3 = 8$; (recall that we have three variables here). These combinations will be 000 through 111 representing the decimal values 0, 1, 2, 3, 4, 5, 6, 7. In the next two columns of the truth table, we compute $X \cdot Y$ and $Y \cdot Z$ for each combination of values of X, Y and Z. Finally, we compute $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$.
The truth table is shown on the next page.

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X Y Z	$X \cdot Y$	$Y \cdot Z$	$(X \cdot Y) \cdot Z$	$X \cdot (Y \cdot Z)$
0 0 0	0	0	0	0
0 0 1	0	0	0	0
0 1 0	0	0	0	0
0 1 1	0	1	0	0
1 0 0	0	0	0	0
1 0 1	0	0	0	0
1 1 0	1	0	0	0
1 1 1	1	1	1	1

Looking at the last two columns of the truth table, we conclude that $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$ are equal for all possible combinations of values of the variables X, Y and Z. Therefore, the conclusion is that equation (1) is valid and theorem (T7') is proven.

<u>Note</u>: You can prove theorem (T7) in a similar way using a truth table.

~~[illegible]~~

<u>Note</u>: The above two and three variable theorems can be found in the text; (table 4-2 on page 199 of text). One more theorem not presented in the text is theorem (T11'') presented below

$$(T11'') \quad X \cdot Y' + Y = X + Y$$

Note: Theorem (T11″) is not related to (T11) and (T11′). I just called it (T11″).

- Proof of theorem (T11″) or proof of $X \cdot Y' + Y = X + Y$

Proof:

$X \cdot Y' + Y = Y + X \cdot Y'$ according to (T6)

$= (Y+X) \cdot (Y+Y')$ according to (T8′)

$= (Y+X) \cdot 1$ according to (T5)

$= Y+X$ according to (T1′)

$= X+Y$ according to (T6).

Another proof of (T11″) is by using a truth table as shown below

X Y	Y'	$X \cdot Y'$	$X \cdot Y' + Y$	$X+Y$
0 0	1	0	0	0
0 1	0	0	1	1
1 0	1	1	1	1
1 1	0	0	1	1

Looking at the last two columns of the truth table, we see that $X \cdot Y' + Y$ and $X+Y$ are equal for all possible combinations of values of the variables X and Y. Therefore, theorem (T11″) holds true.

- Proof of theorem (T8′) or
  $(X+Y)\cdot(X+Z)=X+Y\cdot Z$

<u>Proof</u>

$(X+Y)\cdot(X+Z)=X\cdot(X+Z)+Y\cdot(X+Z)$ according to (T8)
$= X\cdot X+X\cdot Z+Y\cdot X+Y\cdot Z$ according to (T8)
$= X+X\cdot Z+Y\cdot X+Y\cdot Z$ according to (T3′)
$= X+X\cdot Z+X\cdot Y+Y\cdot Z$ according to (T6′)
$= X\cdot 1+X\cdot Z+X\cdot Y+Y\cdot Z$ according to (T1′)
$= X\cdot(1+Z+Y)+Y\cdot Z$ " " (T8)
$= X\cdot 1+Y\cdot Z$ " " (T2)
$= X+Y\cdot Z$ " " (T1′)

- Proof of theorem (T9′) or $X\cdot(X+Y)=X$

<u>Proof</u>

$X\cdot(X+Y)=X\cdot X+X\cdot Y$ according to (T8)
$= X+X\cdot Y$ " " (T3′)
$= X\cdot 1+X\cdot Y$ " " (T1′)
$= X\cdot(1+Y)$ " " (T8)
$= X\cdot 1$ " " (T2)
$= X$ " " (T1′).

- Proof of theorem (T11) or
  $X\cdot Y+X'\cdot Z+Y\cdot Z=X\cdot Y+X'\cdot Z$

<u>Proof</u>

$X\cdot Y+X'\cdot Z+Y\cdot Z=X\cdot Y+X'\cdot Z+1\cdot Y\cdot Z$ according to (T1′)
$= X\cdot Y+X'\cdot Z+(X+X')\cdot Y\cdot Z$ " " (T5)
$= X\cdot Y+X'\cdot Z+X\cdot Y\cdot Z+X'\cdot Y\cdot Z$ " " (T8)

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$= X \cdot Y + X \cdot Y \cdot Z + X' \cdot Z + X' \cdot Y \cdot Z$ according to (T6)
$= X \cdot Y + X \cdot Y \cdot Z + X' \cdot Z + X' \cdot Z \cdot Y$ " " (T6')
" " (T1')
$= X \cdot Y \cdot 1 + X \cdot Y \cdot Z + X' \cdot Z \cdot 1 + X' \cdot Z \cdot Y$ " " (T8)
$= X \cdot Y \cdot (1 + Z) + X' \cdot Z \cdot (1 + Y)$ " " (T2)
$= X \cdot Y \cdot 1 + X' \cdot Z \cdot 1$ " " (T1')
$= X \cdot Y + X' \cdot Z$

Note: I did most of the proofs of the theorems. I leave the remaining as a take home exercise.

Note: Earlier we said that one way of proving theorems with finite number of variables is by using truth tables. I already showed you two examples. What you actually do in this case of using truth tables is evaluating the theorem statement for all possible combinations of the values of the variables. This method is called perfect induction. You can easily prove all the two- and three-variable theorems presented so far by this ~~tech~~ technique.

- Switching-algebra theorems with n variables

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The n-variable theorems follow.

(T12) $X+X+\cdots+X=X$ (Generalized idempotency)

(T12′) $X\cdot X\cdot\cdots\cdot X=X$

(T13) $(X_1\cdot X_2\cdot\cdots\cdot X_n)'=X_1'+X_2'+\cdots+X_n'$ (DeMorgan's theorems)

(T13′) $(X_1+X_2+\cdots+X_n)'=X_1'\cdot X_2'\cdot\cdots\cdot X_n'$

(T14) $[F(X_1,X_2,\cdots,X_n,+,\cdot)]'=F(X_1',X_2',\cdots,X_n',\cdot,+)$ (Generalized DeMorgan's theorem)

(T15) $F(X_1,X_2,\cdots,X_n)=X_1\cdot F(1,X_2,\cdots,X_n)+X_1'\cdot F(0,X_2,\cdots,X_n)$

(T15′) $F(X_1,X_2,\cdots,X_n)=[X_1+F(0,X_2,\cdots,X_n)]\cdot[X_1'+F(1,X_2,\cdots,X_n)]$

* The theorems (T15) and (T15′) are the Shannon's expansion theorems.

Note: As we said before, the theorems with n variables can be proved with the finite induction technique explained on page 6 of this handout.

- Proof of theorem (T13) or proof of $(X_1\cdot X_2\cdot\cdots\cdot X_n)'=X_1'+X_2'+\cdots+X_n'$

Proof: I will prove theorem (T13) using the finite induction technique. I'll first prove that the theorem is true for $n=2$ or I'll prove that $(X_1\cdot X_2)'=X_1'+X_2'$ (2). I'll prove equation (2) using a truth table. This is shown on the next page.

X_1 X_2	$X_1 \cdot X_2$	$(X_1 \cdot X_2)'$	X_1'	X_2'	$X_1'+X_2'$
0 0	0	1	1	1	1
0 1	0	1	1	0	1
1 0	0	1	0	1	1
1 1	1	0	0	0	0

Looking at the 3rd and 6th column of the above truth table, we see that $(X_1 \cdot X_2)'$ and $X_1'+X_2'$ are equal for all possible combinations of values of the variables X_1 and X_2. Therefore equation (2) is true and theorem (T13) holds true for $n=2$.

Assume now that theorem (T13) is true for $n=i$, or assume that

$$(X_1 \cdot X_2 \cdot \cdots \cdot X_i)' = X_1' + X_2' + \cdots + X_i' \quad (3).$$

We need to prove that the theorem is also true for $n=i+1$ or we need to prove that

$$(X_1 \cdot X_2 \cdot \cdots \cdot X_i \cdot X_{i+1})' = X_1' + X_2' + \cdots + X_i' + X_{i+1}'$$

But $(X_1 \cdot X_2 \cdots X_i \cdot X_{i+1})' = [(X_1 \cdot X_2 \cdot \cdots \cdot X_i) \cdot X_{i+1}]' =$

$= (X_1 \cdot X_2 \cdots X_i)' + X_{i+1}'$ according to (2)

$= X_1' + X_2' + \cdots + X_i' + X_{i+1}'$ according to (3).

The proof is now completed and the theorem is true for all values of n.

Note: You can prove theorem (T13$'$) using the finite induction technique. Do it as a homework problem.

Note: Theorem (T14) (the Generalized DeMorgan's theorem) says that given any n-variable logic expression, its complement can be obtained by swapping $+$ and $\cdot$ and complementing all variables. Some examples follow

Example: Find the complement of $(A'+B)\cdot C'$

Answer: $[(A'+B)\cdot C']' = (A'+B)'+(C')' = (A')'\cdot B'+(C')'$
$= A\cdot B'+C$. Here I used theorem (T4) also

Example: Find the complement of
$(A\cdot B'+C)\cdot D'+E$

Answer: $[(A\cdot B'+C)\cdot D'+E]' = [(A\cdot B'+C)\cdot D']'\cdot E' =$
$= [(A\cdot B'+C)'+(D')']\cdot E' = [(A\cdot B')'\cdot C'+D]\cdot E' =$
$= [[A'+(B')']\cdot C'+D]\cdot E' = [(A'+B)\cdot C'+D]\cdot E'$

Here I also used theorem (T4) which states $(X')'=X$.

Example: Find the complement of $A'\cdot B+A\cdot B'$

Answer: $(A'\cdot B+A\cdot B')' = (A'\cdot B)'\cdot(A\cdot B')' =$
$= [(A')'+B']\cdot[A'+(B')'] = (A+B')\cdot(A'+B) =$
$= A\cdot A'+A\cdot B+B'\cdot A'+B'\cdot B = 0+A\cdot B+A'\cdot B'+0 =$
$= A\cdot B+A'\cdot B'$.

Here I also used some other theorems except DeMorgan's. Which are they?

Note: Coming back to DeMorgan's theorems (T13) and (T13′) these theorems simply state:

- The complement of the logical product is the logical sum of the complements.
- The complement of the logical sum is the logical product of the complements.

Another way to view theorems (T13) and (T13′) is the following:

- What theorem (T13) says is that an n-input AND gate whose output is complemented is equivalent to an n-input OR gate whose inputs are complemented. This is shown below by Figures 1 and 2 for $n = 3$.

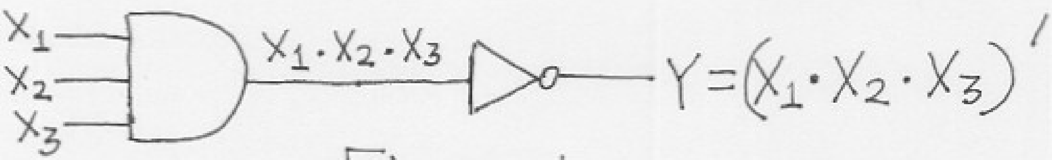

Figure 1

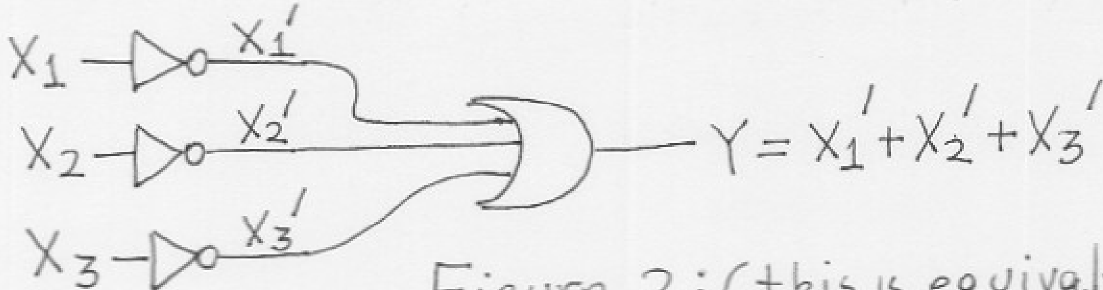

Figure 2; (this is equivalent to Fig 1).

- What theorem (T13′) says is that a n-input OR gate whose output is complemented is equivalent to an n-input AND gate whose inputs are complemented. This is shown by Figures 3 and 4 for $n = 3$ shown on next page.

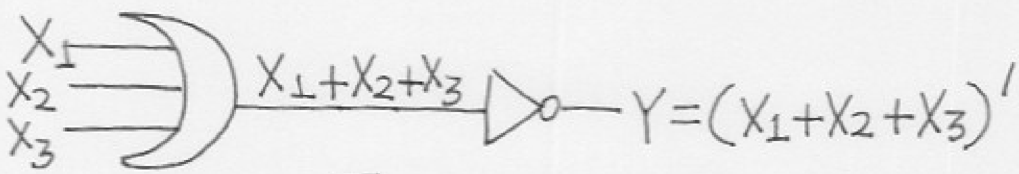

Figure 3

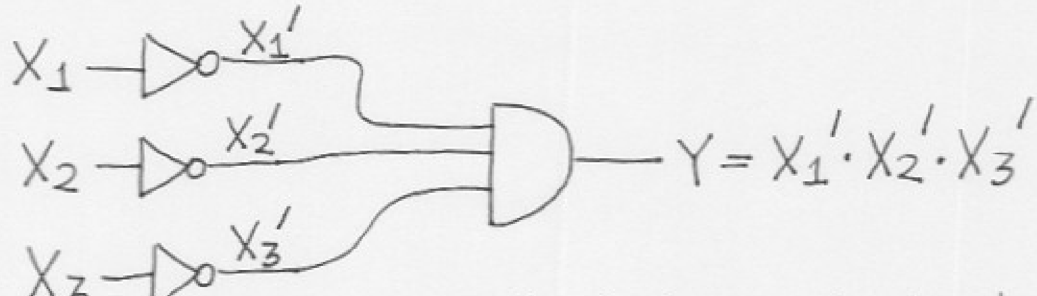

Figure 4; (this is equivalent to Fig. 3)

- <u>Simplification of logic expressions using the presented theorems.</u>

Some examples of simplification of logic expressions using the presented theorems follow.

<u>Example:</u> Simplify $Z = A' \cdot B \cdot C + A'$

<u>Answer:</u>

$$
\begin{aligned}
Z &= A' \cdot B \cdot C + A' \cdot 1 && \text{according to } (T1') \\
&= A'(B \cdot C + 1) && \text{" " } (T8) \\
&= A' \cdot 1 && \text{" " } (T2) \\
&= A' && \text{" " } (T1')
\end{aligned}
$$

<u>Example:</u> Simplify

$$Z = [A + B' \cdot C + D + E \cdot F] \cdot [A + B' \cdot C + (D + E \cdot F)']$$

<u>Answer:</u>

Let X be $X = A + B' \cdot C$ and let Y be $Y = D + E \cdot F$. Then Z becomes $Z = (X + Y) \cdot (X + Y') = X$ according to $(T10')$.

Therefore $Z = X = A + B' \cdot C$

Example: Simplify

$Z=(B'\cdot D+C'\cdot E')\cdot(A\cdot B+C)+(A\cdot B+C)'$

Answer:

$$Z=(B'\cdot D+C'\cdot E')\cdot(A\cdot B+C)+(A\cdot B+C)'$$

Substituting: $Z=\underbrace{\quad}_{X}\cdot\underbrace{\quad}_{Y'}+\underbrace{\quad}_{Y}=X+Y$

according to (T11''); (see page 11 for (T11''))

So $Z=X+Y=B'\cdot D+C'\cdot E'+(A\cdot B+C)'$.

Note that in this example we let $Y=(A\cdot B+C)'$ rather than $A\cdot B+C$ in order to match the form of theorem (T11'').

Note: More examples of simplification of logic expressions using the presented theorems might be offered in the next handout.