EE 2720

Handout #3

- ## Diminished Radix-Complement system

We first present the general case for any radix r.
Consider an n-digit integer number in radix r.

$X=(x_{n-1}x_{n-2}\cdots x_1x_0)_r$

The additive inverse of X which is −X is represented by the so called r−1's complement of X.
The r−1's complement of X is defined to be

$$\boxed{r\text{−1's complement of } X = r^n - 1 - X}$$

Example: Find the 9's complement of the number 1849_{10}.

Answer: Here $r=10$ and $n=4$; (n is the number of digits of the number).
So

9's complement of 1849 $= 10^4 - 1 - 1849 =$
$10{,}000 - 1 - 1849 = 9999 - 1849 = 8150$.

Example: Find the 9's complement of the number 2067_{10}.

Answer: Here $r = 10$ and $n = 4$. So
9's complement of 2067 $= 10^4 - 1 - 2067 =$
$10{,}000 - 1 - 2067 = 9999 - 2067 = 7932$.

Example: Find the 9's complement of the number 8151_{10}.

Answer: Here $r = 10$ and $n = 4$. So
9's complement of 8151 $= 10^4 - 1 - 8151 =$
$10{,}000 - 1 - 8151 = 9999 - 8151 = 1848$.

Example: Find the 9s' complement of the number 0_{10}.

Answer: Here $r = 10$ and $n = 4$. So
9s' complement of 0 $= 10^4 - 1 - 0 =$
$10{,}000 - 1 - 0 = 9999 - 0 = 9999$

- Another way of finding the $r-1$'s complement of a number X is the following:

Consider an n-digit integer number X in radix r.

$$X = (x_{n-1}x_{n-2}\cdots x_1x_0)_r$$

The $r-1$'s complement of X is then

$$\boxed{\begin{array}{l} r-1\text{'s complement of } X = x'_{n-1}x'_{n-2}\cdots x'_1x'_0 \\ \text{where in the above} \\ x'_i = r-1-x_i\,,\quad i=0,1,\ldots,n-1 \end{array}} \qquad ①$$

<u>Example:</u> Find the 9's complement of the number 1849_{10}.

<u>Answer:</u> Here $r=10$, so $r-1=9$. The digit 1 becomes $9-1=8$, the digit 8 becomes $9-8=1$, the digit 4 becomes $9-4=5$ and the digit 9 becomes $9-9=0$. Therefore

9's complement of $1849 = 8150$.

Example: Find the 9's complement of the number 2067_{10}.

Answer: Here $r=10$, so $r-1=9$. The digit 2 becomes $9-2=7$, the digit 0 becomes $9-0=9$, the digit 6 becomes $9-6=3$ and the digit 7 becomes $9-7=2$. Therefore

9's complement of $2067 = 7932$.

Example: Find the 9's complement of the number 8151_{10}.

Answer: Here $r=10$, so $r-1=9$. The digit 8 becomes $9-8=1$, the digit 1 becomes $9-1=8$, the digit 5 becomes $9-5=4$ and the digit 1 becomes $9-1=8$. Therefore

9's complement of $8151 = 1848$

- The Ones'-Complement System

If the radix is $r=2$ in which case we have binary numbers, the diminished radix-complement system is called

ones'-complement (1s' complement) system

Consider an n-bit integer binary number X

$$X = (x_{n-1} x_{n-2} \cdots x_1 x_0)_2$$

Here, the left most bit x_{n-1} is the sign bit and dictates if the number X is positive or negative. If $x_{n-1}=0$ that means that X is positive while if $x_{n-1}=1$ that means that X is negative.

In the ones'-complement system any positive number X ($X>0$) is represented as

$$X = \underset{\substack{\uparrow \\ \text{sign} \\ \text{bit}}}{0}\, x_{n-2} x_{n-3} \cdots x_1 x_0$$

The additive inverse of X is then represented as

$$\boxed{\begin{aligned} -X &= \text{ones'-complement of } X = \\ &= \underset{\substack{\uparrow \\ \text{sign bit}}}{1}\, x'_{n-2} x'_{n-3} \cdots x'_1 x'_0 \end{aligned}} \qquad ②$$

where in the above

$$\boxed{\begin{array}{l} x_i' = 1 \ \text{if} \ x_i = 0 \\ \text{and} \ x_i' = 0 \ \text{if} \ x_i = 1 \ ; \ i = 0, 1, \ldots, n-2 \end{array}} \quad ③$$

The above equations ② and ③ can be derived from equation ① on page 3 of this handout if we consider that here the radix is $r=2$ and each bit x_i can take values of either 0 or 1. Observe that $r-1=2-1=1$ and $1-1=0$ while $1-0=1$.

In general (for the ones'-complement system) if X is

$$X = \underset{\substack{\uparrow \\ \text{sign bit}}}{x_{n-1}} x_{n-2} \cdots x_1 x_0$$

then the additive inverse of X is $-X$ provided on the next page

$$\boxed{\begin{array}{l} -X = \text{ones'-complement of } X = \\ = x'_{n-1}\, x'_{n-2} \cdots x'_1\, x'_0 \end{array}} \qquad \textcircled{4}$$

where in the above

$$\boxed{\begin{array}{l} x'_i = 1 \ \text{ if } \ x_i = 0 \\ \text{and } x'_i = 0 \ \text{ if } \ x_i = 1 \ ; \ i = 0, 1, \dots, n-1 \end{array}} \qquad \textcircled{5}$$

Again equations ④, ⑤ can be derived from equation ① of page 3 as already explained.

- What equations ④ and ⑤ dictate is that in order to obtain the ones'-complement of a number X all we have to do is to complement the individual bits of X ; (this is to say change the zero-bits into ones and the one-bits into zeroes).

The Dynamic Range (DR) of an n-bit integer binary ones'-complement system is

$$DR = [-(2^{n-1}-1) \quad +(2^{n-1}-1)]$$

In the ones'-complement system the number zero has two representations. Just see that

$+0 = 000\cdots000$
$-0 = 111\cdots111$

Lemma: Let X be an n-bit integer signed binary number where the ones'-complement system is used for representing signed numbers. The number X is $X = x_{n-1}x_{n-2}\cdots x_1x_0$. Here x_{n-1} is the sign bit of X. The value of X is then given by

$$\boxed{X_{value} = -(2^{n-1}-1) \times x_{n-1} + \sum_{i=0}^{n-2} x_i \times 2^i} \quad ⑥$$

Equation ⑥ of the lemma simply says that the decimal equivalent of an n-bit integer binary ones'-complement number can be computed the same way as for an unsigned number, except that the sign bit x_{n-1} has a weight of $-(2^{n-1}-1)$ instead of $+2^{n-1}$.

Example: Find the value of the following ones'-complement number: 11101110_2.
Answer: Here $n=8$; (n is the number of bits of the number). Therefore $2^{n-1}-1 = 2^{8-1}-1 = 2^7-1 = 128-1 = 127$.
According to equation ⑥ of the lemma we have $11101110 = -(2^7-1)\times 1 + 2^6\times 1 + 2^5\times 1 + 2^4\times 0 + 2^3\times 1 + 2^2\times 1 + 2^1\times 1 + 2^0\times 0 = -127+64+32+8+4+2 =$
$= -127+110 = -17_{10}$.

Some examples of finding the ones'-complement of 8-bit numbers are presented below

Example: Find the ones'-complement of the number X where $X = 00010001_2 = 17_{10}$.
Answer: On next page ⟶

$$17_{10} = 00010001_2$$

$$\downarrow \text{ complement bits}$$

$$11101110_2 = -17_{10}$$

Example: Find the ones'-complement of the number X where $X = 10011100_2 = -99_{10}$

Answer:

$$-99_{10} = 10011100_2$$

$$\downarrow \text{ complement bits}$$

$$01100011_2 = 99_{10}$$

Example: Find the ones'-complement of the number X where $X = 01110111_2 = 119_{10}$.

Answer:

$$119_{10} = 01110111_2$$

$$\downarrow \text{ complement bits}$$

$$10001000_2 = -119_{10}$$

- Addition/Subtraction in the ones'-complement system

- Addition in the ones'-complement system

Consider two n-bit integer signed binary numbers X and Y where the ones'-complement system is used for representing signed numbers. The numbers X and Y are shown below

$$X = \underset{\uparrow}{x_{n-1}} x_{n-2} \cdots x_1 x_0$$

sign bit

$$Y = \underset{\uparrow}{y_{n-1}} y_{n-2} \cdots y_1 y_0$$

sign bit

- In order to compute X+Y using the ones'-complement system, we add X and Y by ordinary binary addition (by now you know how to do this) and if there is an overal carry-out of the addition with value carry-out=1, add this carry-out back to the result; (this means add 1 to the result).
- This is called end-around carry.

The procedure for ones'-complement addition is shown below

Addition ①

$$\begin{array}{rl} & X = x_{n-1}x_{n-2}\cdots x_1x_0 \\ + & Y = y_{n-1}y_{n-2}\cdots y_1y_0 \\ \hline c & \quad z_{n-1}z_{n-2}\cdots z_1z_0 \end{array}$$

c is overal carry out of addition ①

Addition ②

$$\begin{array}{rr} & z_{n-1}z_{n-2}\cdots z_1z_0 \\ + & c \\ \hline & \underbrace{w_{n-1}w_{n-2}\cdots w_1w_0}_{X+Y} \end{array}$$

Note: In the above if $c=1$ then addition ② must take place. Otherwise if $c=0$ addition ② is not needed.

Example: Using the ones'-complement system perform $X+Y$ where $X = 1110_2 = -1_{10}$ and $Y = 0100_2 = +4_{10}$.

Answer:

$$\begin{array}{r} 1110 \\ +\ 0100 \\ \hline 10010 \end{array}$$

1 is overal carry out of addition and must be added back to the result.

$$\begin{array}{r} 0010 \\ +\qquad 1 \\ \hline 0011 \end{array}$$ adding carry out

$\to X+Y = 0011_2 =$
$= +3_{10}$

Example: Using the ones'-complement system perform $X+Y$ where $X = 0100_2 = +4_{10}$ and $Y = 1000_2 = -7_{10}$

Answer:

$$\begin{array}{r} 0100 \\ +\ 1000 \\ \hline 1100 \end{array}$$

$\to X+Y = 1100_2 = -3_{10}$.

Here overal carry out is 0.

Note: For the ones'-complement system, the addition of a number and its ones'-complement produces as a result the number -0.

The following example explains the above issue

Example: Using the ones'-complement system perform $X+Y$ where $X=0111_2=+7_{10}$ and $Y=1000_2=-7_{10}$.

Answer:

$$\begin{array}{r} 0111 \\ +\,1000 \\ \hline \underbrace{1111} \end{array}$$

$$\hookrightarrow X+Y=1111_2=-0.$$

Here overal carry out is 0.

Note: For the ones'-complement system an addition operation can produce result $+0$ only if the two numbers that are added are $+0$.

Note: The above described procedure for ones'-complement addition produces the correct result as long as this result is within the Dynamic Range (DR) of the system.

- Subtraction in the ones'-complement system

Consider two numbers X and Y. To perform X−Y in the ones'-complement system all we do is

$X-Y = X +$ ones'-complement of Y (end-around overal carry out)

Example: Using the ones'-complement system perform X−Y where $X = 1110_2 = -1_{10}$ and $Y = 1011_2 = -4_{10}$.

Answer: The ones'-complement of Y is ones'-complement of $(1011) = 0100$

We now have

$$\begin{array}{r} 1110 \\ +\ 0100 \\ \hline 10010 \end{array}$$

$\uparrow$

1 is overal carry out of addition and must be added back to the result

$$\begin{array}{r} 0010 \\ +\qquad 1 \\ \hline \underbrace{0011} \end{array}$$ adding carry out

$\hookrightarrow X-Y=0011_2=+3_{10}.$

Example: Using the ones'-complement system perform $X-Y$ where $X=0101_2=+5_{10}$ and $Y=0110_2=+6_{10}$.

Answer: The ones'-complement of Y is ones'-complement of Y = ones'-complement of(0110) = 1001. We now have

$$\begin{array}{r} 0101 \\ +\ 1001 \\ \hline \underbrace{1110} \end{array}$$

$\longrightarrow X-Y=1110_2=-1_{10}.$

Here overal carry out is 0.

<u>Note</u>: The described procedure for ones'-complement subtraction produces the correct result as long as this result is within the Dynamic Range (DR) of the system.

- <u>Overflow/Underflow</u>

For the ones'-complement system the definitions of overflow and underflow are the same as these for the two's-complement system; (see page 23 of Handout #2).

<u>Reminder</u>: The Dynamic Range (DR) on an n-bit integer (binary) ones'-complement system is

$$DR = [-(2^{n-1}-1) \quad +(2^{n-1}-1)].$$

Therefore,

- if result of +/- $> 2^{n-1}-1$ this means overflow

and

- if result of +/- $< -(2^{n-1}-1)$ this means underflow.

Note: An overflow or underflow might occur only when adding numbers of the same sign. When adding numbers with different signs, overflow or underflow can never occur.

• Overflow/Underflow Detection in the ones'-complement system

The detection of overflow and underflow in the ones'-complement system is the same as in the two's-complement system.

— If two positive numbers (having sign bits of 0) added together result in a negative result (having sign bit of 1), this indicates an overflow.

— And if two negative numbers (having sign bits of 1) added together produce a positive result (having sign bit of 0), this indicates an underflow.

Therefore, in the ones'-complement system, in order to detect overflows/underflows we compare the sign bits of the two numbers that are added with the sign bit of the result.

Example: Using the ones'-complement system perform the addition of the 4-bit numbers $X = 0101_2 = +5_{10}$ and $Y = 0011_2 = +3_{10}$.

Answer:

$$\begin{array}{r l} 0101 & \leftarrow \text{positive number} \\ +\,0011 & \leftarrow \text{positive number} \\ \hline 1000 & \leftarrow \text{negative result; (wrong).} \end{array}$$

↳ sign bit = 1. This means that we got a negative result; (observe that the obtained result is $1000_2 = -7_{10} < 0$).

Here an overflow occured.

Remember that the Dynamic Range (DR) of a 4-bit integer ones'-complement system is $DR = [-(2^{4-1}-1) \quad +(2^{4-1}-1)] = [-(2^3-1) \quad +(2^3-1)] = [-7 \quad +7]$ and $X+Y = +5+3 = +8 > +7$ which implies overflow.

Example: Using the ones'-complement system perform the addition of the 4-bit numbers $X = 1010_2 = -5_{10}$ and $Y = 1011_2 = -4_{10}$.

Answer:

 1010 ← ~~positive~~ negative number
+1011 ← negative number
10101

1 is overal carry out of addition and must be added back to the result.

 0101
+   1 adding carry out
 0110 ← positive result; (wrong).

→ sign bit=0. This means that we got a positive result; (observe that the obtained result is $0110_2 = +6_{10} > 0$). Here an underflow occured.

As we said before, the Dynamic Range(DR) of a 4-bit integer ones'-complement system is $DR = [-7 \quad +7]$ and $X+Y = (-5)+(-4) = -9 < -7$ which implies underflow.

- Note: As a homework problem look at Table 2-6 on page 40 of the text. The table shows the decimal numbers in the range from -8 to +7 and their two's-complement, ones'-complement and signed-magnitude representations.

• Addition/Subtraction in the Signed-Magnitude system

When doing additions/subtractions in the sign~~ed~~ magnitude system we basically have to add or subtract the magnitudes of the two numbers and make sure that we return the correct sign bit of the result. There are two cases: The case of a true addition and the case of a true subtraction. A true addition will take place when adding numbers of the same sign or subtracting numbers of different signs. Otherwise a true subtraction will take place. Since signed-magnitude additions/subtractions involve additions/subtractions between unsigned magnitudes, we have to show how we can do additions/subtractions between unsigned numbers.

• Addition between unsigned numbers

You already know how to do this.

Just perform the regular binary addition between the two numbers and look at the overal carry out of the addition to determine if an overflow occured ; (look on pages 21 and 22 of Handout #1 for overflow).

- <u>Subtraction between unsigned numbers</u>

Subtraction between two unsigned numbers might result in a negative result in which case we need a sign bit; (sign indicator).

Consider two n-bit integer binary unsigned numbers X and Y where

$$X = \underset{\substack{\uparrow \\ \text{MSB}}}{x_{n-1}} x_{n-2} \cdots x_1 \underset{\substack{\uparrow \\ \text{LSB}}}{x_0} \qquad \text{and}$$

$$Y = \underset{\substack{\uparrow \\ \text{MSB}}}{y_{n-1}} y_{n-2} \cdots y_1 \underset{\substack{\uparrow \\ \text{LSB}}}{y_0}$$

In order to compute X−Y we have to perform the binary addition

X + (two's-complement of Y).

Suppose that the binary addition $X+$(two's-complement of Y) results in the binary vector

$$(c\, z_{n-1} z_{n-2} \cdots z_1 z_0)_2$$

c is overal
carry out of
addition

- If $c=1$ that means that result $=$ $= X-Y \geqslant 0$ (or $X \geqslant Y$). In this case $X-Y = z_{n-1} z_{n-2} \cdots z_1 z_0$.
- If $c=0$ that means that result$=$ $X-Y < 0$ (or $X<Y$). In this case the result $X-Y$ must have a negative $(-)$ sign while the magnitude of the difference will be two's-complement of $(z_{n-1} z_{n-2} \cdots z_1 z_0)$

In other words, if $c=0$ then

$X-Y=$

$= -$(two's-complement of $(z_{n-1} z_{n-2} \cdots z_1 z_0)$).

__Example:__ Perform the subtraction $X-Y$ where X and Y are the following 4-bit unsigned numbers: $X = 0111_2 = 7_{10}$ and $Y = 0101_2 = 5_{10}$.

__Answer:__ The two's-complement of Y is two's-complement of $(0101) = 1011$

We now have

$$\begin{array}{r} 0111 \\ +\ 1011 \\ \hline 10010 \end{array}$$

↳ since $c=1$ it means result $\geqslant 0$ or

$7-5 = 0010_2 = 2_{10}$

__Example:__ Perform the subtraction $X-Y$ where X and Y are the following 4-bit unsigned numbers: $X = 0101_2 = 5_{10}$ and $Y = 0111_2 = 7_{10}$.

__Answer:__ The two's-complement of Y is two's-complement of $(0111) = 1001$.

We now have

$$\begin{array}{r} 0101 \\ +\ 1001 \\ \hline 01110 \end{array}$$

↳ since $c=0$ it means result <0.

Therefore $5-7 = -(\text{two's-complement of } (1110)) =$

[illegible]

$= -(0010)_2 = -2_{10}.$

Example: Perform the subtraction $X-Y$ where X and Y are the following 4-bit unsigned numbers: $X = 0101_2 = 5_{10}$ and $Y = 0101_2 = 5_{10}.$

Answer: The two's-complement of Y is two's-complement of $(0101) = 1011.$

We now have

$$\begin{array}{r} 0101 \\ +\ 1011 \\ \hline 10000 \end{array}$$

↳ since $c=1$ it means result ≥ 0 or $5-5 = (0000)_2 = 0_{10}.$

Example: Perform the addition $X+Y$ where X and Y are the following 5-bit signed-magnitude numbers: $X = 01100_2 = +12_{10}$ and $Y = 11111_2 = -15_{10}.$

Answer: Here the two numbers X and Y are of different signs and the addition

X+Y needs to be performed. We thus have to perform the following subtraction:

(magnitude of X) − (magnitude of Y) =

= (1100) − (1111) =

= (1100) + (two's-complement of (1111)) =

= (1100) + (0001).

We now have

$$\begin{array}{r} 1100 \\ +\ 0001 \\ \hline 01101 \end{array}$$

↳ since c=0 it means result < 0 or (magnitude of X) − (magnitude of Y) < 0 or magnitude of X < magnitude of Y.

Therefore

- sign bit of result X+Y should be the sign bit of the number with the largest magnitude = sign bit of Y = 1

  and
- magnitude of X+Y =

  = two's-complement of (1101) = 0011.

Therefore $X+Y = 10011_2 = -3_{10}$.

- <u>Note</u>: In case of subtraction between unsigned numbers, an overflow can never occur.